Logic from Language

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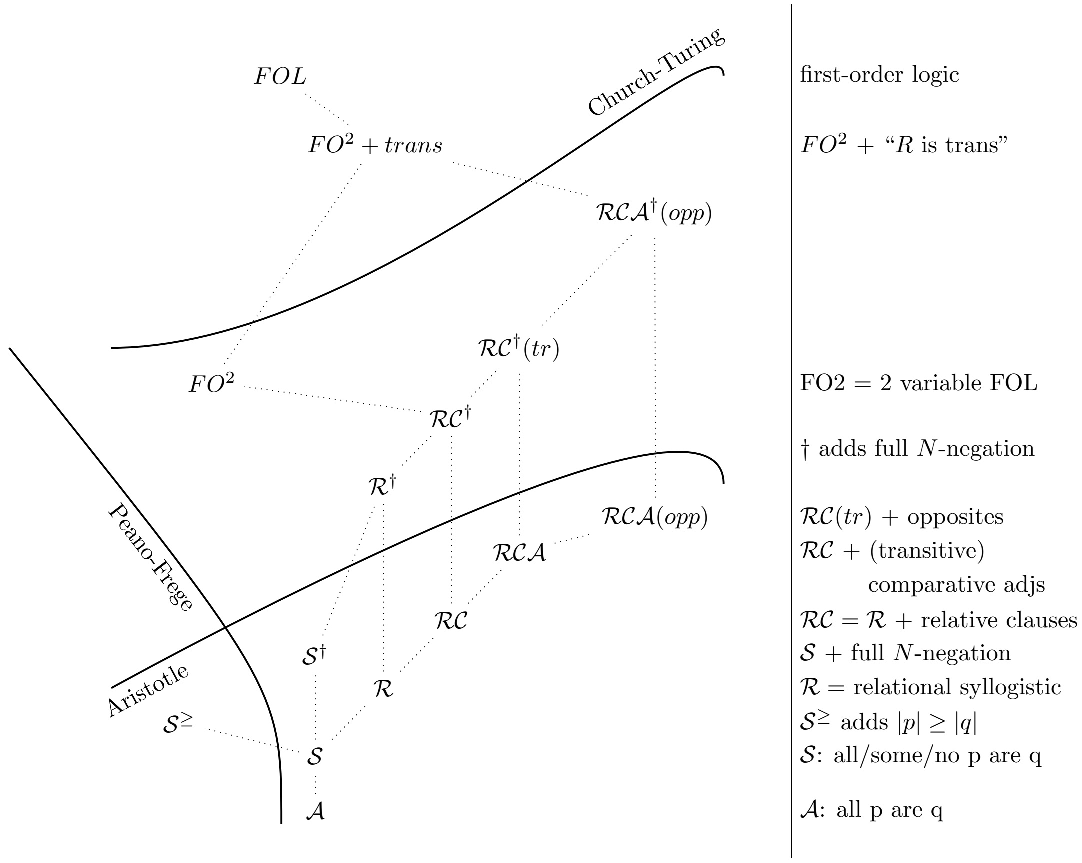
August 2014

stylistic points: the numbering uses roman numbers in theorem-environments, and the exercises should be numbered in the chapters.



Contents

1 Introduction

1.1 Examples of inferences treated

For future reference, it might be useful to list some examples of the kind of phenomena that will interest us.

$$\frac{\textsf{Some dog sees some cat}}{\textsf{Some cat is seen by some dog}} \tag{1.1}$$

$$\frac{\textsf{Bao is seen and heard by every student} \quad \textsf{Amina is a student}}{\textsf{Amina sees Bao}} \tag{1.2}$$

$$\frac{\textsf{All skunks are mammals}}{\textsf{All who fear all who respect all skunks fear all who respect all mammals}} \tag{1.3}$$

$$\begin{array}{l} \textsf{Everyone likes everyone who likes Pat} \\ \underline{\textsf{Pat likes every clarinetist}} \\ \textsf{Everyone likes everyone who likes everyone who likes every clarinetist} \end{array} \tag{1.4}$$

$$\begin{array}{l} \textsf{Every giraffe is taller than every gnu} \\ \textsf{Some gnu is taller than every lion} \\ \underline{\textsf{Some lion is taller than some zebra}} \\ \textsf{Every giraffe is taller than some zebra} \end{array} \tag{1.5}$$

$$\frac{\textsf{More students than professors run} \quad \textsf{More professors than deans run}}{\textsf{More students than deans run}} \tag{1.6}$$

$$\begin{array}{l} \textsf{At most as many xenophobics as yodelers are zookeepers} \\ \textsf{At most as many zookeepers as alcoholics are yodelers} \\ \textsf{At most as many yodelers as xenophobics are alcoholics} \\ \hline \textsf{At most as many zookeepers as alcoholics are xenophobics} \end{array} \tag{1.7}$$

I take all of these to be valid inferences in the sense that a competent speaker who accepts the premisses (above the line) will accept the conclusion. (1.1) involves the passive, as does (1.2). The latter also has conjunction in the VP. (1.3) is a complicated example of iterated subject relative clauses. In my experience with this example during talks, most people by far cannot see that (1.3) is a valid inference. I mention this to point out that fragments which are syntactically very simple might still host non-trivial inferences. Another example that's hard to see is (1.4). To see it, let X be the people who like every clarinetist. By the second premise of (1.4), Pat is an X. From this, it follows that everyone who likes all Xs likes Pat. In informal notation,

$$\text{likes every } X \subseteq \text{likes Pat.}$$

And from what we saw in Problem 3,

$$\text{likes everyone who likes Pat} \subseteq \text{likes everyone who likes every } X.$$

By the first premise of (1.4), everyone belongs to the set on the left. And so everyone belongs to the set on the right, too. This is the conclusion of (1.4).

(1.5) is of interest because people can work out that it is valid, especially if they draw a picture. The key point is that is taller than is transitive: if x is taller than y, and y is taller than z, then x is taller than z. This transitivity is shared with other comparative adjective phrases, and for the simplest of these one might even take it to be a semantic universal. (In more complex phrases, transitivity is lost. For example, is liked by more people than is not transitive, as Condorcet famously noted in his essay on voting of 1785.)

Inference (1.6) is of interest because it is not expressible in *first-order logic.* The same goes for (1.7), and this last inference is harder to see.

2 $\mathcal{A}$: the logic of All p are q

We start with the smallest syllogistic logic "of all", the system $\mathcal{A}$.

2.1 Syntax and Semantics

For the syntax, we start with a collection $\mathbf{P}$ of *atoms*[1] (for nouns). The elements of $\mathbf{P}$ may be anything, and $\mathbf{P}$ might be finite or infinite. We write the atoms as p, q, $\ldots$; occasionally we use subscripts or other devices. We take as *sentences* the expressions

$$\mathsf{All}\ p\ \mathsf{are}\ q$$

where p and q are any atoms in $\mathbf{P}$. There is nothing else in the language, not even boolean connectives. We call this language $\mathcal{A}$[2].

The semantics is based on *models*. A model $\mathcal{M}$ for this fragment $\mathcal{A}$ is a structure

$$\mathcal{M} \quad = \quad (\mathcal{M}, [\![\ \ ]\!])$$

consisting of a set M, together with an *interpretation* $[\![p]\!] \subseteq M$ for each atom $p \in \mathbf{P}$. The main semantic definition is *truth in a model*:

$$\mathcal{M} \models \mathsf{All}\ p\ \mathsf{are}\ q \qquad \text{iff} \qquad [\![p]\!] \subseteq [\![q]\!]$$

We read this in various ways, such as $\mathcal{M}$ *satisfies* All p are q, or All p are q is *true in* $\mathcal{M}$.

Example 2.1 Here is an example of how all of this works. Suppose that $\mathbf{P} = \{n, p, q\}$. In this case, $\mathcal{A}$ would have exactly nine sentences.

Let $M = \{1, 2, 3, 4, 5\}$. Let $[\![n]\!] = \emptyset$, $[\![p]\!] = \{1, 3, 4\}$, and $[\![q]\!] = \{1, 3\}$. This is all we need to specify a model. We'll call this model $\mathcal{M}$. The following sentences are true in $\mathcal{M}$: All n are n, All n are p, All n are q, All p are p, All q are p, and All q are q. (In the first two of these example sentences, we use the fact that *the empty set is a subset of every set*[3].) The other three sentences in $\mathcal{A}$ are false in $\mathcal{M}$.

[1] In Section section-AARC and in Chapter 5 and onward, we'll also have *binary atoms* to represent transitive verbs and other things. So what we are calling *atoms* here will later be re-named to *unary* atoms.

[2] $\mathcal{A}$ stands for "all." Note that $\mathcal{A}$ is really a family of languages, one for each set $\mathbf{P}$ of atoms at the outset. This dependence on primitives is true for practically all logical systems. For most of the work in these notes, we suppress mention of the primitive syntactic items because it makes the notation lighter and because we rarely need to call attention to the primitives in the first place.

[3] This means that our semantics commits us to a view of universal quantification that does not build in existential import. If one wishes to go the other way (in larger logical systems), then it is possible to do so. Either alternative leads to a well-behaved proof system.

We say that $\mathcal{M} \models \Gamma$ iff $\mathcal{M} \models \varphi$ for every $\varphi \in \Gamma$.

We say that $\Gamma \models \varphi$ iff for all $\mathcal{M}$: if $\mathcal{M} \models \Gamma$, then also $\mathcal{M} \models \varphi$.
We read this as Γ *logically implies* φ, or Γ *semantically implies* φ, or that φ *is a semantic consequence of* Γ.

Figure 2.1: Definitions of semantic consequence in any logical system. In these, Γ is a set of sentences in the system, and $\mathcal{M}$ is a model for it.

As soon as we have a formal semantics for a logical system, we get two further notions. These are used with every logic in this book, and so we emphasize their importance by putting them in a figure of their own, Figure 2.1.

Throughout this book, we use Γ (the upper-case Greek letter gamma) to denote a set of sentences in whatever logical system we happen to be discussing. So in this chapter, it denotes a set of sentences in $\mathcal{A}$. We sometimes call Γ a set of *assumptions*. We also sometimes call it a *theory*, following the usage in modern logic.

Example 2.2 Here is an example of a semantic consequence which can be expressed in $\mathcal{A}$: We claim that

$$\{\mathsf{All}\ p\ \mathsf{are}\ q, \mathsf{All}\ n\ \mathsf{are}\ p\} \models \mathsf{All}\ n\ \mathsf{are}\ q. \tag{2.1}$$

To see this, we give a straightforward mathematical proof.[4] Let $\mathcal{M}$ be any model for $\mathcal{A}$, assuming that the underlying set $\mathbf{P}$ contains n, p, and q. Assume that $\mathcal{M}$ satisfies All p are q and All n are p. We must prove that $\mathcal{M}$ also satisfies All n are q. From our first assumption, $[\![p]\!] \subseteq [\![q]\!]$. From our second, $[\![n]\!] \subseteq [\![p]\!]$. It is a general fact about sets that the inclusion relation (written as $\subseteq$ here) is *transitive*, and so we conclude that $[\![n]\!] \subseteq [\![q]\!]$. This verifies that indeed $\mathcal{M}$ satisfies All n are q. And since $\mathcal{M}$ was arbitrary, we are done.

Incidentally, when Γ is a finite set that is written out explicitly on the left side of a $\models$ or $\vdash$ symbol, we usually omit it from the notation. So instead of writing (2.1), we would usually write

$$\mathsf{All}\ p\ \mathsf{are}\ q, \mathsf{All}\ n\ \mathsf{are}\ p \models \mathsf{All}\ n\ \mathsf{are}\ q.$$

Example 2.3 Next, we have an example of a failure of semantic consequence:

$$\mathsf{All}\ p\ \mathsf{are}\ q \not\models \mathsf{All}\ q\ \mathsf{are}\ p.$$

[4] There is no real need to use a formal system to do mathematical proofs! They are much more readable when done informally.

To show that a given set Γ does not logically entail another sentence φ, we need to build a model $\mathcal{M}$ of Γ which is not a model of φ. In this example, Γ is $\{\textsf{All } p \textsf{ are } q\}$, and φ is $\textsf{All } q \textsf{ are } p$. We can get a model $\mathcal{M}$ that does the trick by setting $M = \{1, 2\}$, $[\![p]\!] = \{1\}$, and $[\![q]\!] = \{1, 2\}$. For that matter, we could also use a different model, say $\mathcal{N}$, defined by $N = \{61\}$, $[\![p]\!] = \emptyset$, and $[\![q]\!] = \{61\}$.

Example 2.4 Here is a more complicated example. At this point, we present it mainly as a challenge.

$$\Gamma \;=\; \left\{ \begin{array}{lll} \textsf{All } j \textsf{ are } k, & \textsf{All } j \textsf{ are } l, & \textsf{All } k \textsf{ are } l, \\ \textsf{All } l \textsf{ are } k, & \textsf{All } l \textsf{ are } m, & \textsf{All } k \textsf{ are } n, \\ \textsf{All } m \textsf{ are } q, & \textsf{All } p \textsf{ are } q, & \textsf{All } q \textsf{ are } p \end{array} \right\}$$

True or false: $\Gamma \models \textsf{All } p \textsf{ are } n$? If true, give a reason. If false, give a model of Γ where $[\![p]\!] \not\subseteq [\![n]\!]$.

The point here not to simply solve this particular problem, but to give an *algorithm* which would solve all problems of this type, and to prove your answer. We'll do this in Section 2.5.

2.2 Proof theory

Figure 2.2 presents a *proof system* for the language $\mathcal{A}$. The system has two *inference rules.* Each rule allows conclusions to be inferred from some set of premises. The conclusion of a rule is the sentence below the horizontal line, and the premise(s) are above the line.

Let us discuss the second rule, (Barbara) first. The rule says that All p are q may be inferred from the two premises All p are n and All n are q. That is, a node in a tree may be labeled All p are q provided it has two nodes above it in the tree, one labeled All p are n for some atom n, and the other labeled All n are q.

The first rule, named (Axiom), says that All p are p may label a node in a proof tree provided that node have *no nodes above it.* So no premises are required, and none may appear.

The name (Barbara) comes from medieval logic. The reason for this is that logicians used A for universal assertions. (Of course, they did not use English, so it is purely a coincidence that A stands for *All.*) The rule has three As. Classical logicians interpolated consonants "randomly" to get the names of their rules, hence bArbArA. Our rule (Axiom) is so-named because an axiom in a proof system is like a rule of inference with no premises.

Definition Let Γ be a set of sentences in $\mathcal{A}$. A *proof tree over* Γ is a finite tree[5] $\mathcal{T}$

$$\dfrac{}{\mathsf{All}\ p\ \mathsf{are}\ p}\ \textsc{Axiom} \qquad\qquad \dfrac{\mathsf{All}\ p\ \mathsf{are}\ n \quad \mathsf{All}\ n\ \mathsf{are}\ q}{\mathsf{All}\ p\ \mathsf{are}\ q}\ \textsc{Barbara}$$

Figure 2.2: The logical system for $\mathcal{A}$.

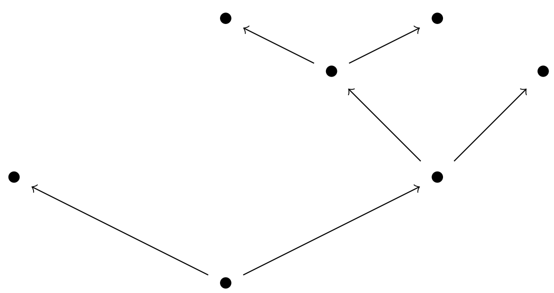
whose nodes are labeled with sentences, and each node is either a leaf node labeled with an element of Γ, or else matches one of the rules in the proof system in Figure 2.2.

$\Gamma \vdash \varphi$ means that there is a proof tree $\mathcal{T}$ for over Γ whose root is labeled φ. We read this as Γ *proves* φ, or Γ *derives* φ, or that φ *follows in our proof system from* Γ.

Example 2.5 Here is an example, chosen to make several points: Let Γ be

$$\{\mathsf{All}\ l\ \mathsf{are}\ m, \mathsf{All}\ q\ \mathsf{are}\ l, \mathsf{All}\ m\ \mathsf{are}\ p, \mathsf{All}\ n\ \mathsf{are}\ p, \mathsf{All}\ l\ \mathsf{are}\ q\}$$

Let φ be $\mathsf{All}\ q\ \mathsf{are}\ p$. Here is a proof tree showing that $\Gamma \vdash \varphi$: We start with the tree



(2.2)

and then we label the various points with sentences in $\mathcal{A}$. Rather than give the function in any explicit way, we merely illustrate the way it works. We take

$$\dfrac{\mathsf{All}\ q\ \mathsf{are}\ l \qquad \dfrac{\dfrac{\mathsf{All}\ l\ \mathsf{are}\ m \quad \dfrac{}{\mathsf{All}\ m\ \mathsf{are}\ m}\ \textsc{Axiom}}{\mathsf{All}\ l\ \mathsf{are}\ m}\ \textsc{Barbara} \qquad \mathsf{All}\ m\ \mathsf{are}\ p}{\mathsf{All}\ l\ \mathsf{are}\ p}\ \textsc{Barbara}}{\mathsf{All}\ q\ \mathsf{are}\ p}\ \textsc{Barbara} \tag{2.3}$$

Again, the proof tree (technically speaking) is a tree together with a labeling function taking the nodes of the tree to sentences. But we never need to be so explicit, and it will be better to think of the tree as it appears in (2.3). Similarly, whenever we refer to the *root* and *leaves* of the tree, we really mean the sentences that label the root and leaves.

Let us check that what we have in (2.4) really is a proof tree according to our definition. All of the leaves belong to Γ except for one: that is $\mathsf{All}\ m\ \mathsf{are}\ m$. This last leaf matches the first rule in Figure 2.2. That is, this rule allows us to use $\mathsf{All}\ m\ \mathsf{are}\ m$ with no

[5] See page 16 for more on trees.

Suppose we are given a logical language and a semantics for it, defining a notion $\Gamma \models \varphi$. Suppose we also have a proof system for the same language, defining a notion $\Gamma \vdash \varphi$.

The proof system is *sound for the semantics* if whenever $\Gamma \vdash \varphi$, we also have $\Gamma \models \varphi$. The proof system is *complete for the semantics* if whenever $\Gamma \models \varphi$, we also have $\Gamma \vdash \varphi$.

Figure 2.3: Soundness and Completeness of a proof system relative to a semantics.

parents. All of the other nodes in the tree match the second rule. They all have two parents, and they are special cases of the rule (BARBARA).

Derivation trees come with the information of which rule is used at the various nodes of the tree, the way we do it in (2.3). But frequently we drop all the rules. So our derivation tree would look like

$$\dfrac{\textsf{All } q \textsf{ are } l \qquad \dfrac{\dfrac{\textsf{All } l \textsf{ are } m \quad \textsf{All } m \textsf{ are } m}{\textsf{All } l \textsf{ are } m} \qquad \textsf{All } m \textsf{ are } p}{\textsf{All } l \textsf{ are } p}}{\textsf{All } q \textsf{ are } p} \tag{2.4}$$

Finally, note also that some sentences from Γ are not used as leaves. This is permitted according to our definition. Also, there is a smaller proof tree that also shows that $\Gamma \vdash \varphi$: we could drop All m are m (The reason why have the rule (AXIOM) is so that that we can have one-element trees labeled with sentences of the form All l are l.)

We have a proof system for the language $\mathcal{A}$ of this chapter. Earlier, we defined the semantics of this language, using models. The main technical work is to connect the semantics and the proof system. The main definitions pertaining to this connection are found in Figure 2.3. Like much of the work in this chapter, these definitions will be used in all of the logical system we shall see.

Our next goal is to prove the soundness of the proof system with respect to the semantics. This proof uses *induction*, and so we turn to that topic. To prove something about all proof trees over a set Γ, we use *induction on proof trees over* Γ[6]. The statement that we need may be found in Figure 2.4.

We'll see examples of induction on proof trees in Proposition 2.2.1 and in Exercises 5 and 6.

Proposition 2.2.1 (Soundness) *If* $\Gamma \vdash \varphi$, *then* $\Gamma \models \varphi$.

Proof We prove this by induction on proof trees over Γ as stated in Figure 2.4. We take $S(\mathcal{T})$ to be the assertion:

$$\text{if the root of } \mathcal{T} \text{ is labeled } \varphi\text{, then every } \mathcal{M} \models \Gamma \text{ also satisfies } \varphi \tag{2.5}$$

[6] If you don't know about induction, you might wish to consult some other sources to learn about induction in other contexts, especially *induction on numbers*. Proofs by induction will only play a minor role in what we do, so if you don't mind taking a few results on faith, it would be possible to read on forgetting induction completely. However, the settings that we have are simple enough that you might even be able to pick up induction from the rest of the book.

Let $S(\mathcal{T})$ be an assertion about proof trees. Suppose that

(i) $S(\mathcal{T})$ whenever $\mathcal{T}$ is a one-point tree labeled with a sentence from Γ.

(ii) $S(\mathcal{T})$ whenever $\mathcal{T}$ is a proof tree over Γ whose root is justified by (Axiom).

(iii) Let $\mathcal{T}$ be a proof tree over Γ whose root is justified by (Barbara), and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the subtrees right above the root. If $S(\mathcal{T}_1)$ and also $S(\mathcal{T}_2)$, then $S(\mathcal{T})$.

Assuming these conditions, we have $S(\mathcal{T})$ for all proof trees $\mathcal{T}$ over Γ.

Figure 2.4: The induction principle for proof trees over Γ.

First, we show that $S(\mathcal{T})$ when $\mathcal{T}$ is a one-point tree labeled with a sentence from Γ. In this case, φ belongs to Γ. And so every model $\mathcal{M}$ of *all* sentences in Γ is a fortiori model of φ.

Second, we show that $S(\mathcal{T})$ when $\mathcal{T}$ is proof tree over Γ whose root is justified by (Axiom). In this case, the root of $\mathcal{T}$ is a sentence of the form All p are p. (Also, in this case, the entire tree is the root. But this is irrelevant.) Every sentence All p are p is true in all models $\mathcal{M}$ whatsoever, regardless of whether $\mathcal{M}$ satisfies Γ or not. So $S(\mathcal{T})$ holds in this case.

Now assume that the root of $\mathcal{T}$ is All n are q, and right above this we have two trees whose roots are All n are p and All p are q, respectively. Assume that every model of Γ satisfies All n are p, and also that every model of Γ satisfies All p are q. Fix a model $\mathcal{M}$ of Γ. Then in $\mathcal{M}$, $[\![n]\!] \subseteq [\![p]\!]$, and also $[\![p]\!] \subseteq [\![q]\!]$. Continuing to think about this fixed model, we also have $[\![n]\!] \subseteq [\![q]\!]$. But $\mathcal{M}$ was an arbitrary model of Γ. So we have shown than every model of Γ satisfies the root of $\mathcal{T}$. That is, $S(\mathcal{T})$ holds.

The paragraphs above show that for all proof trees $\mathcal{T}$ over Γ, every model of Γ satisfies the root of $\mathcal{T}$. Put another way, if $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$. ⊣

Remark The soundness proofs of all the logical systems in this book are all pretty much the same as the one in Proposition 2.2.1. They are always inductions, and the crux of the matter is usually a simple fact about sets. (In Proposition 2.2.1, the crux of the matter is that the inclusion relation $\subseteq$ on subsets of a given set is always a transitive relation.) We almost never present the soundness proof in any detail.

Proposition 2.2.1 tells us that the formal logical system for $\mathcal{A}$ is not going to give us any bad results. (This is what *soundness* means.) Now this is a fairly weak point. If we dropped some of the rules, it would still hold. Even if we decided to be conservative and say that $\Gamma \vdash \varphi$ *never* holds, Proposition 2.2.1 would still hold. So the more interesting question to ask is whether the logical system is *strong* enough to prove everything it *should prove*. We want to know if $\Gamma \models \varphi$ implies that $\Gamma \vdash \varphi$. If this implication does hold for all Γ and φ, then we say that our system is *complete*.

2.3 Preliminaries: graphs and preorders

At this point, we have presented a language called $\mathcal{A}$, a semantics for it, and a sound proof system to go along with the semantics. The main theoretical work is to show that the system is complete: if $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$. We also must discuss the algorithmic properties of the system. That is, if someone gives us a (finite) set Γ and a sentence φ, how can we determine whether or not $\Gamma \vdash \varphi$? We shall study these soon, but first we need some preliminary notions from general mathematics.

Graphs A *graph* is a pair $\mathcal{G} = (G, \to)$, where G is a set and $\to$ is a relation on G. The elements of G are called *nodes*, *vertices*, or *points*. If g and h are nodes of G, we usually write $g \to h$ to mean that g and h are related by the relation $\to$. But sometimes the style of exposition dictates other notation, such as $(g, h) \in \to$. Alternately, we might say that g is the *parent* of h. The relation is also called the *edge relation* of the graph. There is no requirement on this relation: G might have *loops* (i.e., we might have $g \to g$ for some, or even all nodes of G). But a graph need not have any loops at all. (For that matter, a graph need not have *any* nodes at all! The empty graph is perfectly fine for us. But as with all similar situations, other authors may differ on this point of usage.)

Example 2.6 Here is a graph:

$$\begin{tikzcd} a \arrow[r] & b \arrow[r] & c & d \arrow[loop right] & & e \end{tikzcd} \tag{2.6}$$

Technically, we have drawn a picture of a graph $\mathcal{G}$, where the node set G is $\{a, b, c, d, e\}$, and the edge relation $\to$ is $\{(a, b), (b, c), (d, d)\}$.

Example 2.7 For each set Γ of sentences of $\mathcal{A}$ and each set G of atoms, we get a graph $\mathcal{G}_{\Gamma,G}$. The nodes of $\mathcal{G}_{\Gamma,G}$ are the elements of the given set G, and the relation $\to$ is defined by

$$x \xrightarrow[\Gamma]{} y \quad \text{iff} \quad \text{the sentence } \mathsf{All}\ x\ \mathsf{are}\ y \text{ belongs to } \Gamma$$

We call this the *all-graph of* Γ *on* G. Graphs of this form are the main reason why we introduce graphs in this section.

There are two special cases of this construction which interest us. One is where $G = \mathbf{P}$, the set of all atoms. The second is where we have a set Γ and another sentence φ, and G is the set of all atoms which occur in Γ or in φ.

Definition If $\mathcal{G}$ is any graph, we write $g \xrightarrow{*} h$ to mean that there is a finite *path* from g to h following the edge relation in the graph. The relation $\xrightarrow{*}$ is called the *reflexive-transitive closure* of the graph.

For example, in the graph $\mathcal{G}$ in Example 2.6, we have $a \xrightarrow{*} c$, using the path $a \to b \to c$. But paths can be of length 1 or even 0. The full listing of $\xrightarrow{*}$ is

$$a \xrightarrow{*} b \quad b \xrightarrow{*} c \quad a \xrightarrow{*} c \quad a \xrightarrow{*} a \quad b \xrightarrow{*} b \quad c \xrightarrow{*} c \quad d \xrightarrow{*} d \quad e \xrightarrow{*} e$$

Trees We defined proof trees for our logic in Section 2.2. But we neglected at that point to say what *trees* actually are. A tree is a graph with a designated node called the *root* with the property that every node is reachable from the root by a unique path. A *leaf* of a tree is a node with no successors (no outgoing edges).

Preorders A *preorder* is a pair $\mathbb{P} = (P \leq)$, where P is a set, and $\leq$ is a relation on P which is both *reflexive* and *transitive*. Reflexivity means that $p \leq p$ for all $p \in P$. Transitivity means that if $p \leq q$ and $q \leq r$, then also $p \leq r$.

Perhaps the most natural example of a preorder would be the *power set preorder* on a set X. This is $(\mathcal{P}(X), \subseteq)$, where $\mathcal{P}(X)$ is the set of subsets of X, and $\subseteq$ is the *inclusion relation* on $\mathcal{P}(X)$: $A \subseteq B$ means that every element of A is also an element of B.

There is a natural way of turning every graph $\mathcal{G}$ into a preorder $\mathbb{P}$.

Proposition 2.3.1 *For any graph* $(\mathcal{G}, \to)$*, the structure* $(\mathcal{G}, \xrightarrow{*})$ *is a preorder.*

Proof The fact that we allow paths to have length 0 tells us that for each point g in $\mathcal{G}$, we have $g \xrightarrow{*} g$. Thus, $\xrightarrow{*}$ is reflexive. For the transitivity, suppose that $g \xrightarrow{*} h$ and $h \xrightarrow{*} i$. Then there is a path from g and h, and another path from h to i. Chaining the paths gives a path from g to i, showing that $g \xrightarrow{*} i$. $\dashv$

2.4 Completeness

The last section was a digression from our main thread in this chapter, the logical system for $\mathcal{A}$. We now return to that thread and prove the completeness of the system.

Definition Let Γ be a set of sentences in any fragment containing All. Define $u \leq_\Gamma v$ to mean that

$$\Gamma \vdash \textsf{All } u \textsf{ are } v. \tag{2.7}$$

As always, we simplify the notation by dropping the subscript Γ if it is clear from the context.

Proposition 2.4.1 *For all sets* Γ*,* $(\mathbf{P}, \leq_\Gamma)$ *is a preorder.*

Proof To check that $\leq$ is reflexive, let $u \in \mathbf{P}$. Then we have a one-line proof of All u are u. Indeed, this is the point of having this kind of no-premise rule in the proof system. For the transitivity, we put together a proof of $u \leq v$ with a proof of $v \leq w$ to show that $u \leq w$. $\dashv$

We are going to the fact that $\leq$ is a preorder frequently in this book, and often without explicitly mentioning it.

The canonical model of a set Γ. We now have an important model construction. Starting with a set $\mathbf{P}$ of atoms and a theory Γ in our language $\mathcal{A}$, we build a model $\mathcal{M}$ *from the syntax and the proof theory.*

$$\begin{array}{lllll} M & = & \mathbf{P} & & \\ [\![u]\!] & = & \downarrow u & = & \{v \in M : v \leq u\} \end{array} \tag{2.8}$$

We read $\downarrow u$ as "the down-set of u." So we are taking the nouns to be the elements of our model, and we interpret a given noun u as the set of all nouns v which we can prove to be included in u. $\mathcal{M}$ is called the *canonical model of* Γ.

Lemma 2.4.2 *The canonical model of* Γ *satisfies* Γ*.*

Proof Suppose All p are q belongs to Γ. Then $p \leq q$. We must show that $[\![p]\!] \subseteq [\![q]\!]$. Let $v \in [\![p]\!]$. Then by (2.8), $v \leq p$. Since $\leq$ is transitive, $v \leq q$. That is, $v \in [\![q]\!]$. This for all v shows our result. $\dashv$

Theorem 2.4.3 *The logic of Figure 2.2 is complete for $\mathcal{A}$: If $\Gamma \models$ All p are q, then $\Gamma \vdash$ All p are q.*

Proof Suppose that $\Gamma \models$ All p are q. We must prove that $\Gamma \vdash$ All p are q. Consider $\mathcal{M}$ from (2.8). By Lemma 2.4.2, $\mathcal{M}$ satisfies all sentences in Γ. By hypothesis, we see that All p are q is true in $\mathcal{M}$. Thus $[\![p]\!] \subseteq [\![q]\!]$. We always have $p \leq p$, and so $p \in [\![p]\!]$. Hence $p \in [\![q]\!]$. This means that $p \leq q$, so that $\Gamma \vdash$ All p are q, as desired. $\dashv$

Remark The proof in fact shows that if $\Gamma \not\models$ All p are q, then All p are q is false in $\mathcal{M}$. So we have a very special and important fact: to see whether a given sentence φ follows from Γ or not, we only have to see whether φ is true or false in one model, $\mathcal{M}$. We say that this model $\mathcal{M}$ is a *characteristic model of* Γ.

2.5 Algorithmic analysis

The original definition of the entailment relation $\Gamma \models \varphi$ involves looking at *all* models of the language. If we are given Γ and φ and we want to know whether or not $\Gamma \models \varphi$, we can say "no" by producing a *counter-model*: a model of Γ where φ fails. If we want to say "yes", the easiest way would be to provide a derivation in our proof system of φ from Γ. This would show that $\Gamma \vdash \varphi$, and then by soundness (Proposition 2.2.1), we would know that indeed $\Gamma \models \varphi$.

But suppose that we are given Γ and φ, and we don't know whether or not $\Gamma \models \varphi$. For example, suppose we are faced with Γ and φ from Example 2.4. What happens now?

We could try building a counter-model and searching for a proof at the same time. This would work, but we would like something better. That "something better" would be an algorithm that was more detailed and more organized than a blind search, and that either gave a derivation or a counter-model. This is the topic of this section. The centerpiece of this section is Theorem 2.5.1 just below, a *refined version* of the completeness argument which we saw in Section 2.4. Theorem 2.5.1 leads to an algorithm which we present in Figure 2.5.

Theorem 2.5.1 *Let Γ be any set of sentences in $\mathcal{A}$, let G be any set of atoms which includes the atoms in Γ, and let $\mathcal{G}$ be the all-graph of Γ on G. Let $\overset{*}{\underset{\Gamma}{\Rightarrow}}$ be the reflexive-transitive closure of the graph relation in $\mathcal{G}$. Let $p, q \in G$. Then the following are equivalent:*

(i) $p \overset{*}{\underset{\Gamma}{\Rightarrow}} q$.

(ii) $\Gamma \vdash$ All p are q.

(iii) $\Gamma \models$ All p are q.

Proof (1)$\Rightarrow$(2): we show by induction on the number n that if q is reachable from p in $\mathcal{G}$ by a path of length n, then $\Gamma \vdash$ All p are q. If $n = 0$, then $p = q$, and we have a one-point proof tree of All p are p. Suppose our result is true for n, and also that q is reachable from p in $\mathcal{G}$ by a path of length $n+1$. Fix such a path. Suppose that q' is the point on the path right before q. Then q' is reachable from p in $\mathcal{G}$ by a path of length n. So by induction hypothesis, we have $\Gamma \vdash$ All p are q'. Since there is an edge from q' to q, we know that the sentence All q' are q belongs to Γ. So we can take $\mathcal{T}$ and add a step at the end:

$$\dfrac{\begin{matrix}\vdots \\ \textsf{All } p \textsf{ are } q'\end{matrix} \quad \textsf{All } q' \textsf{ are } q}{\textsf{All } p \textsf{ are } q}\ \textsc{Barbara}$$

This is a proof tree over Γ, and it shows that $\Gamma \vdash$ All p are q.

(2)$\Rightarrow$(3): this is the soundness result for the logic which we saw in Proposition 2.2.1.

(3)$\Rightarrow$(1): we show the contrapositive. Assume that $p \overset{*}{\underset{\Gamma}{\not\Rightarrow}} q$. Let $\mathcal{N}$ be the following model: the universe is the set G, and the interpretation is given by

$$[\![u]\!] \quad = \quad \{v \in G : v \overset{*}{\underset{\Gamma}{\Rightarrow}} u\} \tag{2.9}$$

We claim that $\mathcal{N} \models \Gamma$. To see this, suppose that Γ contains the sentence All x are y. We show that $[\![x]\!] \subseteq [\![y]\!]$. Let $z \in [\![x]\!]$. That is, $z \in G$ and there is a path from z to x in $\mathcal{G}_{\Gamma,G}$. The atom y occurs in Γ and hence belongs to G. Then taking the path from above and adding the edge from x to y shows that $z \overset{*}{\underset{\Gamma}{\Rightarrow}} y$. This for all x shows that indeed $[\![x]\!] \subseteq [\![y]\!]$; and the preceding observation for all sentences in Γ shows that $\mathcal{N} \models \Gamma$. By definition of $\overset{*}{\underset{\Gamma}{\Rightarrow}}$, $p \overset{*}{\underset{\Gamma}{\Rightarrow}} p$. Also, $p \in G$ by hypothesis. So $p \in [\![p]\!]$. Our overall assumption that $p \overset{*}{\underset{\Gamma}{\not\Rightarrow}} q$ tells us that $p \notin [\![q]\!]$. Hence $\mathcal{N} \not\models$ All p are q.

This completes the proof. $\dashv$

We are given a finite set Γ and a sentence $\varphi = \textsf{All } p \textsf{ are } q$, and we wish to tell whether or not $\Gamma \vdash \varphi$. If $\Gamma \vdash \varphi$, we give a derivation; if $\Gamma \nvdash \varphi$, we give a counter-model.

(i) Let G be the set of atoms in Γ together with p and q.

(ii) Let $\mathcal{G} = (G, \underset{\Gamma}{\to})$ be the all-graph of Γ on G.

(iii) Calculate the reflexive-transitive closure $\overset{*}{\underset{\Gamma}{\to}}$ of the edge relation of $\mathcal{G}$.

(iv) If Step (iii) has shown that $p \overset{*}{\underset{\Gamma}{\to}} q$, then a path in $\mathcal{G}$ gives a proof tree in the logic, showing that $\Gamma \vdash \varphi$. If not, then we construct a model $\mathcal{N}$ of Γ where φ fails. We take the universe to be G, and interpret each atom u by $\{v : v \overset{*}{\underset{\Gamma}{\to}} u\}$.

Figure 2.5: The algorithm to determine if $\Gamma \vdash \varphi$ or not.

Corollary 2.5.2 *Let $\Gamma \cup \{\textsf{All } p \textsf{ are } q\}$ be a set of sentences in $\mathcal{A}$. Let G be the set of atoms in Γ together with p and q. Let $\mathcal{G}$ be the all-graph of Γ on G. Then the following are equivalent:*

(i) $\Gamma \models \varphi$.

(ii) $p \overset{*}{\underset{\Gamma}{\to}} q$ *in* $\mathcal{G}$.

Corollary 2.5.2 leads to the algorithm presented in Figure 2.5. More to the point, Corollary 2.5.2 shows that the algorithm is correct. Given a finite set Γ and a sentence φ (say, $\textsf{All } p \textsf{ are } q$), we first take the set of atoms in Γ. We add p and q to this set if they are not there already, and then the resulting set is called G. We construct the all-graph of Γ on G. The all-graph is directly read from Γ. There is a well-known algorithm which, given any relation R finds the reflexive-transitive closure R^*. So we apply this algorithm to the all-graph of Γ on G, and we ask whether or not $p \overset{*}{\underset{\Gamma}{\to}} q$. If so, then the proof of (1) $\Rightarrow$ (2) in Theorem 2.5.1 shows how to turn the path into a derivation in the logic. If not, then the model $\mathcal{N}$ constructed in (3) $\Rightarrow$ (1) is a *counter-model*: a model of Γ where the putative conclusion $\textsf{All } p \textsf{ are } q$ is false.

Example 2.8 We have seen a theory Γ in Example 2.4. The relevant set G here is $\{j,k,l,m,n,p,q\}$. The all-graph of Γ on G is shown in Figure 2.6. The figure also shows the preorder $\mathbb{P}$ associated to Γ.

We challenged you in Example 2.4 to see whether $\Gamma \vdash \textsf{All } p \textsf{ are } n$ or not. We can see from Figure 2.6 that $p \overset{*}{\not\to} n$. Even more, we can exhibit a counter-model $\mathcal{M}$: We take $M = \{j,k,l,m,n,p,q\}$, and

$$\begin{array}{lllllllll} [\![j]\!] & = & \{j\} & [\![m]\!] & = & \{j,k,l,m\} & [\![p]\!] & = & \{j,k,l,m,p,q\} \\ [\![k]\!] & = & \{j,k,l\} & [\![n]\!] & = & \{j,k,l,n\} & [\![q]\!] & = & \{j,k,l,m,p,q\} \\ [\![l]\!] & = & \{j,k,l\} & & & & & & \end{array}$$

$$\Gamma = \left\{ \begin{array}{lll} \textsf{All } j \textsf{ are } k, & \textsf{All } j \textsf{ are } l, & \textsf{All } k \textsf{ are } l, \\ \textsf{All } l \textsf{ are } k, & \textsf{All } l \textsf{ are } m, & \textsf{All } k \textsf{ are } n, \\ \textsf{All } m \textsf{ are } q, & \textsf{All } p \textsf{ are } q, & \textsf{All } q \textsf{ are } p \end{array} \right\}$$

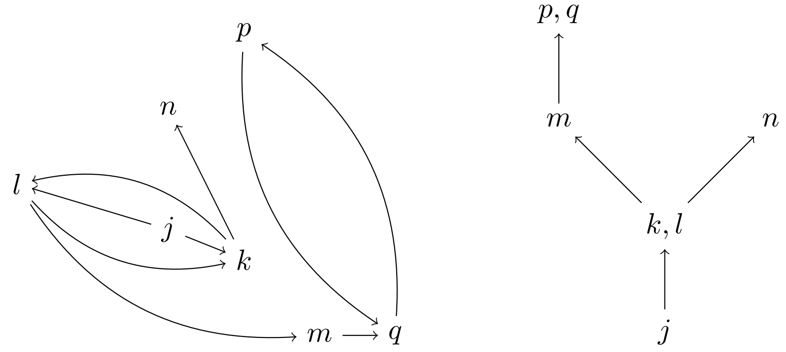


Figure 2.6: Γ is from Example 2.4. We show the all-graph of Γ on the set of atoms which occur in Γ, There is no special reason why we laid out the graph on the page like this; it was "random." We also show the preorder associated to Γ. The preorder is displayed as a *Hasse diagram*: For example, since $j \leq k$, we draw j below k. Note also that $k \leq l$ and $l \leq k$. We indicate this in the picture by situating k and l together.

We got each of these from (2.9): the interpretation of each atom u is the set of atoms which can reach u in the all-graph.

Comments There are a few comments to be made at this point. These are based on questions which I have received in teaching this material, or similar material, to bright students.

(1) *Doesn't this proof confuse syntax and semantics?* This is a good question, since we are building a model out of the "material from the syntax" (in this case, the atoms). But when one thinks about it, there is nothing wrong with building a model out of chairs, numbers, abstract objects, or even the same objects that we used in the syntax. It is more interesting that the interpretation function $[\![\ ]\!]$ in the model was defined in terms of a syntactic notion, the proof system. Again, we are free to define the semantics of a model any way we like. It is *interesting* that we prove completeness in this way. But in a sense it should not be such a surprise. For completeness is about a relation between syntax and semantics, and so it makes sense that it involves a single structure that has aspects of both.

(2) *I thought that the semantic assertion $\Gamma \models \varphi$ meant that all models of Γ are again models of φ. How is it that we only argue completeness on one particular model rather than many models?* It's true that our semantic assertion $\Gamma \models \varphi$ is a statement about *all* models. But this does not mean that in proving something we need to *use* all the

models. In a sense, the question can be turned around to make an observation: the model $\mathcal{M}$ we built from a set Γ is as "bad" as any model could possibly be! It makes true any sentence which does not follow from Γ. So once we have built a single model that covers for all the models, we can use that one model to prove completeness.

What you need to know to go on At this point, you should understand a few things well: the *syntax* of the language in this chapter, beginning with the set $\mathbf{P}$ of atoms; the *semantics* of the language, including the definition of a *model* and the definition of when a model *satisfies* a given sentence; the *proof theory*, defined in terms of two *rules of inference*; the statements of *soundness* and *completeness*; the proof that our logical system is complete (it was done twice); the general issue of *algorithms for proof search in a given logic*; how the algorithm works for the particular logic in this chapter; the definition of a *counter-model* to an assertion of the form $\Gamma \vdash \varphi$.

The reason that you should understand all of this is because in most of the remaining chapters, we shall see an ever-increasing array of logics. Each time, we'll see all of the *iitalicized* points; of course, the specific details will vary with the logic. It would be good to really understand all of the points above. One way to be sure would be to write out explanations of everything mentioned in the last paragraph. Be sure to use your own words, but also to use all of the symbols that we introduced in this chapter.

2.6 Exercises

Exercise 1. Here is an exercise just to see if you got the main points of this chapter. Let

$$\Gamma \quad = \quad \{\textsf{All } x \textsf{ are } y, \textsf{All } z \textsf{ are } y, \textsf{All } y \textsf{ are } w\}.$$

Let φ be $\textsf{All } x \textsf{ are } z$. For each of the following assertions, say whether true or false, with a reason:

(i) $\Gamma \models \varphi$.

(ii) $\Gamma \vdash \varphi$

(iii) $x \leq_\Gamma w$.

(iv) $x \xrightarrow[\Gamma]{} w$.

(v) $x \xrightarrow[\Gamma]{*} w$.

If you are using the soundness or completeness of the proof system for $\mathcal{A}$ in any part, please be sure to note this. That is, please be aware when you are using significant facts!

Exercise 2. Suppose that $\Gamma \vdash \varphi$. For each of the following assertions, tell whether it is True or False, with a reason.

(i) There are infinitely many proof trees over Γ which show that $\Gamma \vdash \varphi$.

(ii) Any two proof trees over Γ which show that $\Gamma \vdash \varphi$ must share a leaf.

(iii) If a sentence of the form All p are p occurs in a derivation tree, then that occurrence must be at a leaf.

Exercise 3. Here is a different way to prove the completeness of the logic for $\mathcal{A}$. Suppose that $\Gamma \models$ All p are q. Consider the model $\mathcal{N}$ where $N = \{*\}$, some one-element set, and

$$[\![u]\!] \quad = \quad \begin{cases} N & \text{if } \Gamma \vdash \textsf{All } p \textsf{ are } u \\ \emptyset & \text{otherwise} \end{cases} \tag{2.10}$$

Please note that in (2.10), the noun p is the same one as in overall statement.

By reasoning as in the proof of Theorem 2.4.3, use $\mathcal{N}$ to show that $\Gamma \vdash$ All p are q.

Exercise 4. This exercise has to do with an aspect of the canonical model defined in (2.8). Recall that we took M to be $\mathbf{P}$, the set of all atoms with underlies our language in this chapter. If we had taken M to be the set of atoms which occur in Γ, would our construction have worked? That is, what (if anything) would have to change in the rest of this section?

Exercise 5. Let Γ be a set of sentences in $\mathcal{A}$. Prove that for all p and q,

$$p \xrightarrow[\Gamma]{*} q \quad \text{iff} \quad p \leq_\Gamma q$$

[Hint: you will need to use induction, in two forms. One direction of this exercise requires induction on numbers; the other uses induction on proofs in our proof system.]

Exercise 6. Suppose that Γ is the empty set of sentences. Under what conditions on p and q will we have $\Gamma \vdash$ All p are q?

(i) Give a semantic reason for your answer, one having to do with models.

(ii) Give a syntactic reason for your answer, one having to do with the proof system. In this part, you will need to use induction on proofs.

Exercise 7. Let Γ be a theory in any logical system extending $\mathcal{A}$. Let $\mathcal{M}$ be any model with the property that

$$[\![\ ]\!] : \mathbf{P} \to \mathcal{P}(M^*)$$

is a monotone function. That is, if $p \leq_\Gamma q$, then $[\![p]\!] \subseteq [\![q]\!]$. Prove that $\mathcal{M} \models$ All p are q, whenever this last sentence belongs to Γ. [This is not a hard exercise, but it is one that will be used all the time.]

Exercise 8. Let $\mathbb{P} = (P, \leq)$ be any preorder, and let $\downarrow$ be the function from P to its power-set $\mathcal{P}(P)$ given by

$$\downarrow p \quad = \quad \{q \in P : q \leq p\}$$

(i) Show that $\downarrow$ is a monotone function.

(ii) The result in part (1) is a generalization of what we saw in Exercise 7 just above. Why is this?

Exercise 9. Here is another logical system. Instead of atoms denoting subsets of a given set, our syntax starts with a set $\mathbf{N}$ of *names.* We use letters like a, b, etc. for names here. Then we take as sentences the expressions $a = b$, where $a, b \in \mathbf{N}$. For the semantics, we start with a set M and interpret a name a by an *element* $[\![a]\!] \in M$. (Again, we are not using subsets they way we did in this chapter.) This gives us the definition of a *model* $\mathcal{M}$.

(i) As soon as we have a semantics, we get notions like $\Gamma \models \varphi$ for this new language. State what $\Gamma \models \varphi$ means, and then give an example of a set Γ and a sentence φ where $\Gamma \models \varphi$ holds, and another example where it does not hold.

(ii) Find a logical system that defines a notion $\Gamma \vdash \varphi$, and give an example of a proof in your system.

(iii) Prove that your system is complete.

Exercise 10. If you have worked Exercise 9, then the next step is to combine your system from that exercise with the logical system for $\mathcal{A}$. We start with sets $\mathbf{P}$ and $\mathbf{N}$ of atoms and names, respectively. In the syntax, we use sentences of the following forms:

$$\begin{array}{l} \textsf{All } p \textsf{ are } q \\ a = b \\ a \textsf{ isa } p \end{array}$$

The semantics is just what you would expect. In particular, in a given model $\mathcal{M}$, we say that a isa p is true if $[\![a]\!] \in [\![p]\!]$. The proof theory for this system should contain the rules in Figure 2.2, the rules in your system from Exercise 9, and the two extra rules below:

$$\frac{a \textsf{ isa } p \quad \textsf{All } p \textsf{ are } q}{a \textsf{ isa } q} \qquad \frac{a = b \quad a \textsf{ isa } p}{b \textsf{ isa } p}$$

Prove the completeness of the system.

3 Additions to $\mathcal{A}$

The logic $\mathcal{A}$ is perhaps the simplest possible logic, both in terms of the syntax and semantics, and also in terms of the proof theory. The simple ideas in Sections 2.4 and **??** get used over and over as we add to the syntax, sometimes with variations.

3.1 Binary intersection terms

This section presents one of those additions, a language $\mathcal{A}_2$ where we can talk about *intersections of two sets.*

Syntax, semantics, and proof theory Recall that we begin with a set $\mathbf{P}$ of atoms. In addition to $\mathbf{P}$, the fragment of this section uses a different (but related) set $\mathbf{P}_2$ whose elements are the formal pairs $x \wedge y$ of elements of $\mathbf{P}$. We call these items *terms.* We allow x and y to be the same, and so we have terms $x \wedge x$ for all atoms x. We take sentences of the form

$$\textsf{All } x \wedge y \textsf{ are } z \wedge w.$$

We could read this as

$$\textsf{All } x \textsf{ which are } y \textsf{ are also } z \textsf{ which are } w.$$

The $\wedge$ symbol is not a term-forming operator in the usual sense; we cannot form terms like $x \wedge (y \wedge z)$. All of the terms are "binary meets". We also don't have variables on their own as terms in this language. So we cannot say All $x \wedge y$ are z. Instead, we would say All $x \wedge y$ are $z \wedge z$. This detail is a choice on our part, and we made it to simplify some of the work below. We could have made other choices for the syntax, and then these other choices would be reflected in different formulations of the proof theory.

We call this language $\mathcal{A}_2$.

Technically, the terms $x \wedge y$ and $y \wedge x$ are different. However, the idea is that they should denote the same set, and so they will be semantically identical. That is, for all models $\mathcal{M}$,

$$[\![x \wedge y]\!] \quad = \quad [\![y \wedge x]\!] \tag{3.1}$$

(We have not given the semantics yet, but you might try to guess it and then to check that (3.1) holds.) Moreover, the proof theory will also identify them. That is, we'll have $\vdash$ All $x \wedge y$ are $y \wedge x$, and $\vdash$ All $y \wedge x$ are $x \wedge y$.

The semantics of $\mathcal{A}_2$ works as follows. Recall that the set $\mathbf{P}_2$ is built from some set $\mathbf{P}$ of atoms. We use models of the form

$$\mathcal{M} \quad = \quad (M, [\![\ \ ]\!] : \mathbf{P} \to \mathcal{P}M)$$

$$\frac{}{\textsf{All } x \wedge y \textsf{ are } y \wedge x} \qquad \frac{\textsf{All } x \wedge y \textsf{ are } z \wedge w \quad \textsf{All } z \wedge w \textsf{ are } a \wedge b}{\textsf{All } x \wedge y \textsf{ are } a \wedge b}$$

$$\frac{}{\textsf{All } x \wedge y \textsf{ are } x \wedge x} \qquad \frac{\textsf{All } x \wedge y \textsf{ are } z \wedge w \quad \textsf{All } x \wedge y \textsf{ are } a \wedge b}{\textsf{All } x \wedge y \textsf{ are } z \wedge a}$$

Figure 3.1: Rules for $\mathcal{A}_2$, the logic of binary intersections.

Then we define truth in such a model by:

$$\mathcal{M} \models \textsf{All } x \wedge y \textsf{ are } z \wedge w \quad \text{iff} \quad [\![x]\!] \cap [\![y]\!] \subseteq [\![z]\!] \cap [\![w]\!].$$

Derived semantic notions Just as with all our logical languages, once we have a definition of $\mathcal{M} \models \varphi$, we get the derived notions and $\mathcal{M} \models \Gamma$ and $\Gamma \models \varphi$ automatically. (See Figure 2.1.) The main technical work is in crafting a proof system so as to define $\Gamma \vdash \varphi$ in a syntactic way, and then to prove the soundness and completeness of this logical system.

Logic We have a logical system for this language, using the rules in Figure 3.1. The first two rules are the preorder property of the order relation. The third law tells us that $\textsf{All } x \wedge y \textsf{ are } x \wedge x$. As another instance of this rule, we get $\textsf{All } x \wedge y \textsf{ are } y \wedge y$. The final law in the figure is a variant on saying that $z \wedge a$ is the greatest lower bound of z and a. It says that if an arbitrary element $(x \wedge y)$ is below both z and a, then it is below $z \wedge a$.

Soundness/Completeness Fix a set Γ of sentences in the logic. We define an order $\leq_\Gamma$ on $\mathbf{P}_2$ by

$$x \wedge y \leq_\Gamma a \wedge b \quad \text{iff} \quad \Gamma \vdash \textsf{All } x \wedge y \textsf{ are } a \wedge b$$

As before, we sometimes drop the Γ from the notation. It is important to note that $\leq_\Gamma$ is always a preorder.

Theorem 3.1.1 *$\Gamma \vdash \textsf{All } x \wedge y \textsf{ are } z \wedge w$ iff $\Gamma \models \textsf{All } x \wedge y \textsf{ are } z \wedge w$.*

Proof The soundness being trivial, we only detail the completeness half of our theorem. Suppose that $\Gamma \models \textsf{All } x \wedge y \textsf{ are } z \wedge w$. Let $\mathcal{M}$ be the structure defined by

$$\begin{array}{rcl} M & = & \mathbf{P}_2 \\ [\![u]\!] & = & \{a \wedge b : \Gamma \vdash \textsf{All } a \wedge b \textsf{ are } u \wedge u\} \end{array}$$

We claim that $\mathcal{M} \models \Gamma$. For this, let $\textsf{All } u \wedge v \textsf{ are } a \wedge b$ belong to Γ. Let $c \wedge d \in [\![u]\!] \cap [\![v]\!]$. From Γ we have $\textsf{All } c \wedge d \textsf{ are } u \wedge u$ and also $\textsf{All } c \wedge d \textsf{ are } v \wedge v$. We have a derivation from Γ:

$$\frac{\dfrac{\begin{array}{c}\vdots\\ \textsf{All } c \wedge d \textsf{ are } u \wedge u\end{array} \quad \begin{array}{c}\vdots\\ \textsf{All } c \wedge d \textsf{ are } v \wedge v\end{array}}{\textsf{All } c \wedge d \textsf{ are } u \wedge v} \qquad \textsf{All } u \wedge v \textsf{ are } u \wedge u}{\textsf{All } c \wedge d \textsf{ are } u \wedge u}$$

So $c \wedge d \in [\![u]\!]$. Similarly, $c \wedge d \in [\![v]\!]$, and so $c \wedge d \in [\![u]\!] \cap [\![v]\!]$. In this way, the sentence $u \wedge v \leq a \wedge b$ holds in $\mathcal{M}$.

At this point, we know that $\mathcal{M} \models \Gamma$. By our overall assumption that $\Gamma \models \textsf{All } x \wedge y \textsf{ are } z \wedge w$, we know that $\textsf{All } x \wedge y \textsf{ are } z \wedge w$ holds in $\mathcal{M}$. Now $x \wedge y \in [\![x]\!] \cap [\![y]\!]$, easily. So $x \wedge y \in [\![z]\!] \cap [\![w]\!]$. Thus we have a derivation from Γ:

$$\dfrac{\begin{matrix}\vdots \\ \textsf{All } x \wedge y \textsf{ are } z \wedge z\end{matrix} \qquad \begin{matrix}\vdots \\ \textsf{All } x \wedge y \textsf{ are } w \wedge w\end{matrix}}{\textsf{All } x \wedge y \textsf{ are } z \wedge w}$$

This completes the proof. ⊣

3.2 Verbs and relative clauses

The subject of this chapter is additions that we can make to the system $\mathcal{A}$ whose sentences are just those of the form All x are y. We saw one addition in Section 3.1, and now we turn to another. In this section, we add verbs. The idea is that we would like to consider sentences such as All cats chase all rats. To put this in the mold of $\mathcal{A}$, we think of this as All cats (chase all rats). So our semantics will arrange that chase all rats is interpreted by a set, just as cats is; and then we shall say that All cats (chase all rats) is true in a given model iff $[\![\textsf{cats}]\!] \subseteq [\![\textsf{chase all rats}]\!]$. And to interpret chase all rats by a set invites us to go further. We can just as well have chase all x where x is "the same type of object as" chase all rats. For example, we could have chase all (see all birds). The idea is that this would denote the entities which chase everything which chases all birds. Now the natural language construct that this brings up are *relative clauses*. In English, these are the snippets of language which frequently (but not always) start with a *relative pronoun* such as *who* or *which* and which modify and follow some *head noun*. Our formal language in this section is not an adequate treatment relative clauses: it is only the beginning. It only covers *subject relative clauses* rather than *object relative clauses*. For example, we'll have sentences corresponding to Everyone who likes all plumbers sits but not Everyone who all plumbers like sits. Worse, we'll have no way to modify head nouns at all. To extend what we do in this section to cover sentences like Every carpenter who likes all plumbers sits would involve mixing the work of this section with that of Section 3.1. So overall, what we have is a formal language which is not going to read like English at all (despite what we would want). It is going to serve as a language in which to interpret a small number of sentences involving verbs and relative clauses. But despite is meager linguistic interest, it gives us an opportunity to build a new (and, we hope, interesting) logical system.

Syntax of $\mathcal{A}(\mathcal{RC})$ We start with one collection $\mathbf{P}$ of *unary atoms* (for nouns), just as we did with $\mathcal{A}$ in Section 2. But we also start with another collection, $\mathbf{R}$ of and another of *binary atoms*. We use these for transitive verbs.

We define the *set terms* of $\mathcal{A}(\mathcal{RC})$ to be the smallest collection containing the unary atoms and with the property that if x is a set term and r is a binary atom, then r all x is a set term. We frequently call these *terms*.

Note that set terms allow recursion. So we get set terms like

$$\mathsf{see\ all\ (like\ all\ (hate\ all\ dogs))}.$$

(Of course, this is on the assumption that **P** and **R** contain the words shown above.) Although we are not yet done with the syntax of the language, the interpretation of the set term above in a given model would be the set of individuals who see all who like all who hate all dogs.

The *sentences* of $\mathcal{A}(\mathcal{RC})$ are the expressions All x y, where x and y are set terms. We frequently use parentheses in the syntax to increase the readability.

Semantics A model $\mathcal{M}$ for $\mathcal{A}(\mathcal{RC})$ is a set M, the "universe", together with interpretations of the atoms. For each unary atom p, we have an interpretation $[\![p]\!] \subseteq M$. And for each binary atom r, we have an interpretation $[\![r]\!] \subseteq M \times M$.

We use an inductive definition to interpret the set terms. The model comes with interpretations of the unary atoms, the "base case" of set terms. And the general case is

$$[\![\mathsf{see\ all}\ x]\!] \quad = \quad \{m \in M : \text{for all } n \in [\![x]\!],\ (m,n) \in [\![\mathsf{see}]\!]\}$$

And then we say

$$\mathcal{M} \models \mathsf{All}\ x\ y \quad \text{iff} \quad [\![x]\!] \subseteq [\![y]\!]$$

Example 3.1 Consider the model $\mathcal{M}$ with universe $\{1,\ldots,7\}$, and with interpretations of three unary atoms, boys, males, $\mathsf{females}$, and one binary atom, see, shown in Figure 3.2.

In this model,

$$\begin{array}{lcl} [\![\mathsf{see\ all\ boys}]\!] & = & \{2,3,4,5,7\} \\ [\![\mathsf{see\ all\ males}]\!] & = & \{3,4,5,7\} \\ [\![\mathsf{see\ all\ females}]\!] & = & \{5\} \\ [\![\mathsf{see\ all\ (see\ all\ boys)}]\!] & = & \{5\} \end{array}$$

Here are some examples of sentences false and true in the model:

$$\begin{array}{l} \mathcal{M} \not\models \mathsf{All\ (see\ all\ boys)\ (see\ all\ males)} \\ \mathcal{M} \models \mathsf{All\ (see\ all\ females)\ (see\ all\ (see\ all\ boys))} \\ \mathcal{M} \models \mathsf{All\ (see\ all\ (see\ all\ boys))\ (see\ all\ females)} \end{array}$$

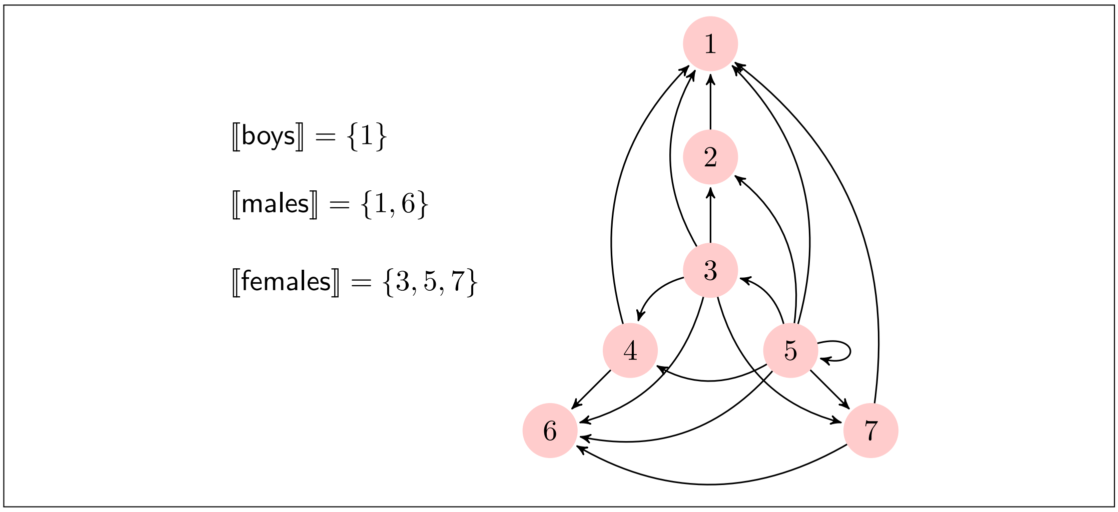


Figure 3.2: The model from Example 3.1.

The logic of our language $\mathcal{A}(\mathcal{RC})$ is presented in Figure 3.3. The main new law is the last one, which we call (ANTI), for *antitone*. To check the soundness, let $\mathcal{M}$ be a model, and assume that $[\![q]\!] \subseteq [\![p]\!]$. To see that $[\![r \text{ all q}]\!] \subseteq [\![r \text{ all p}]\!]$, let $x \in [\![r \text{ all q}]\!]$. Let $y \in [\![p]\!]$. So $y \in [\![q]\!]$ as well. Thus $x[\![r]\!]y$. This for all $y \in [\![p]\!]$ shows that $x \in [\![r \text{ all } p]\!]$.

Example 3.2 Here is a derivation showing that $\text{All } x \text{ (see all } y), \text{All } z\ y \vdash \text{All } x \text{ (see all } z)$:

$$\dfrac{\text{All } x \text{ (see all } y) \qquad \dfrac{\text{All } z\ y}{\text{All (see all } y)\text{(see all } z)}\ \textsc{Anti}}{\text{All } x \text{ (see all } z)}\ \textsc{Barbara}$$

The canonical model $\mathcal{M}_\Gamma$ of a set of assertions Γ in this logic We are given a set Γ in this language $\mathcal{A}(\mathcal{RC})$. We aim to build a model $\mathcal{M} = \mathcal{M}_\Gamma$ of Γ with the property that if $\mathcal{M} \models \varphi$, then $\Gamma \vdash \varphi$. This would prove the completeness theorem for our logic, because if $\Gamma \not\vdash \varphi$, then $\mathcal{M} \not\models \varphi$. (In other words, every sentence which is true in all models of Γ (hence in $\mathcal{M}$) would be provable from Γ.)

Here is the definition. In it, p is a unary atom and r a binary one.

$$\begin{array}{rcl} [\![p]\!] & = & \{x : \Gamma \vdash \text{All } x\ p\} \\ [\![r]\!] & = & \{(x,y) : \Gamma \vdash \text{All } x \text{ } (r \text{ all } y)\} \end{array} \tag{3.2}$$

The clause for unary atoms is what we saw in Section 2. But the clause for the binary atoms should not be obvious.

$$\frac{}{\textsf{All}\ x\ x}\ \textsc{Axiom} \qquad \frac{\textsf{All}\ x\ y \quad \textsf{All}\ y\ z}{\textsf{All}\ x\ z}\ \textsc{Barbara} \qquad \frac{\textsf{All}\ y\ x}{\textsf{All}\ (r\ \textsf{all}\ x)\ (r\ \textsf{all}\ y)}\ \textsc{Anti}$$

Figure 3.3: The logical system for $\mathcal{A}(\mathcal{RC})$. Note that we are using this with x, y, and z as set terms, not only as unary variables.

Example 3.3 Suppose we have two unary atoms p and q, and one binary atom r. Let

$$\Gamma \quad = \quad \{\textsf{All}\ p\ \textsf{are}\ q\}$$

To make the notation simpler, let us write p_0 for p, and p_{n+1} for r $\textsf{all}$ p_n; we adopt similar notation for q. Then in the canonical model of Γ, we have $[\![p]\!] = \{p_0\}$, $[\![q]\!] = \{p_0, q_0\}$, and the interpretation of r is shown below:

$$\begin{tikzcd}
p_0 & p_1 \arrow[l] & p_2 \arrow[l] \arrow[dl] & p_3 \arrow[l] & p_4 \arrow[l] \arrow[dl] & \cdots \\
q_0 & q_1 \arrow[l] \arrow[ul] & q_2 \arrow[l] & q_3 \arrow[l] \arrow[ul] & q_4 \arrow[l] & \cdots
\end{tikzcd}$$

It should be noted that we are cheating a bit in this. It is fairly easy to see that the arrows above are *included* in the canonical model. To do this, we only would need to exhibit the derivations corresponding top the arrows. But to show that no others arrows are present, one would have to do a lot more work. In fact, it is usually easier to offer a semantic proof of this. For that, one would take the model above and then to show that every missing arrow corresponds to a semantic fact (by examining the model more deeply), and then it *follows* that what we have above is the canonical model. If this point is mysterious, then please don't worry: we'll revisit this issue as we proceed.

Lemma 3.2.1 *Let Γ be a set of sentences in $\mathcal{A}(\mathcal{RC})$, and let $\mathcal{M}$ be the canonical model of Γ as defined in (3.2). For all set terms x,*

$$[\![x]\!] \quad = \quad \{y : \Gamma \vdash \textsf{All}\ y\ x\}$$

Proof The proof is by induction on x. When x is a unary atom, our result is by the definition of our model. Assume that x is a set term and that our lemma holds for x; we prove that

$$[\![r\ \textsf{all}\ x]\!] \quad = \quad \{y : \Gamma \vdash \textsf{All}\ y\ (r\ \textsf{all}\ x)\}$$

First, let $y \in [\![r\ \textsf{all}\ x]\!]$. By induction hypothesis, $x \in [\![x]\!]$. So $(y, x) \in [\![r]\!]$. That is, $\Gamma \vdash \textsf{All}\ y\ (r\ \textsf{all}\ x)$. In the other direction, assume that $\Gamma \vdash \textsf{All}\ y\ (r\ \textsf{all}\ x)$. Suppose

We are given a finite set Γ of $\mathcal{A}(\mathcal{RC})$ and a sentence φ, and we wish to tell whether or not $\Gamma \vdash \varphi$. If $\Gamma \vdash \varphi$, we give a derivation; if $\Gamma \not\vdash \varphi$, we give a counter-model.

(i) Let M_0 be the set of set terms in $\Gamma \cup \{\varphi\}$. Let M be the closure of M_0 under subterms. Let M^* be $M \cup \{r \textsf{ all } z : z \in M_1\}$.

(ii) Write $\Gamma \vdash^{\text{confined}} \psi$ if there is a derivation tree of ψ from Γ all of whose set terms belong to $\Gamma \cup \{\varphi\}$ or are subterms of the root, ψ. Then calculate

$$\{(x, y) \in M \times M^* : \Gamma \vdash^{\text{confined}} \textsf{All } x\ y\}.$$

(iii) Let $\mathcal{M}$ be the model whose universe is M, and whose structure is defined as follows:

$$\begin{array}{rcl} [\![x]\!] & = & \{y \in M : \Gamma \vdash^{\text{confined}} \textsf{All } y\ x\} \\ [\![r]\!] & = & \{(x, y) \in M \times M : \Gamma \vdash^{\text{confined}} \textsf{All } x\ (r \textsf{ all } y)\} \end{array}$$

(iv) If Step 2 includes a verification that $\Gamma \vdash \varphi$, then of course we are done. If not, then $\mathcal{M}$ is a model of Γ where φ fails (see Theorem 3.2.7).

Figure 3.4: An algorithm to determine if $\Gamma \vdash \varphi$ or not.

that $z \in [\![x]\!]$. Then by induction hypothesis, $\Gamma \vdash \textsf{All } z\ x$. Now consider the following derivation:

$$\dfrac{\begin{array}{c}\vdots \\ \textsf{All } y\ (r \textsf{ all } x)\end{array} \qquad \dfrac{\begin{array}{c}\vdots \\ \textsf{All } z\ x\end{array}}{\textsf{All } (r \textsf{ all } x)\ (r \textsf{ all } z)}}{\textsf{All } y\ (r \textsf{ all } z)}$$

So $(y, z) \in [\![r]\!]$. This for all $z \in [\![x]\!]$ shows that $y \in [\![r \textsf{ all } x]\!]$. ⊣

Lemma 3.2.2 $\mathcal{M} \models \Gamma$.

Proof Suppose that Γ contains $\textsf{All } u\ v$. To see that this sentence holds in $\mathcal{M}$, let $y \in [\![u]\!]$. By Lemma 3.2.1, $\Gamma \vdash \textsf{All } y\ u$. And then using the logic, we have $\Gamma \vdash \textsf{All } y\ v$. ⊣

Lemma 3.2.3 *If* $\mathcal{M} \models \varphi$*, then* $\Gamma \vdash \varphi$.

Proof Let φ be $\textsf{All } a\ b$. Let $\mathcal{M}$ be the model constructed above. $\mathcal{M} \models \Gamma$, by Lemma 3.2.2. So $[\![a]\!] \subseteq [\![b]\!]$ in $\mathcal{M}$. But $a \in [\![a]\!]$, using (Axiom) in our system and also Lemma 3.2.1. And so $a \in [\![b]\!]$. And this shows that $\Gamma \vdash \textsf{All } a\ b$, as desired. ⊣

Theorem 3.2.4 *The logical system in Figure 3.3 is sound and complete for* $\mathcal{A}(\mathcal{RC})$.

Algorithmic analysis Theorem 3.2.4 shows the completeness of the logic under study. Our next order of business is to do the algorithmic analysis. If we have a *finite* set S of set terms and finite set Γ, and another sentence φ, then we can answer the question

Is there a derivation showing $\Gamma \vdash \varphi$ all of whose set terms belong to S?

We can answer this question efficiently, simply by generating all of the relevant derivations.

However, in general we have Γ and φ, but not S. And so to tell whether or not $\Gamma \vdash \varphi$, we need to know a priori about a relevant set S which is "big enough" for our purposes. The good news is that we can find such a set S. It turns out that we can take S to be all set terms in $\Gamma \cup \{\varphi\}$, together with all subterms of those terms.

Figure 3.4 presents an algorithm to tell whether or not a given finite set Γ logically implies a given sentence φ. The algorithm depends on a few definitions.

Definition If p is a unary atom, the only subterm of p is p itself. The subterms of r all x are the subterms of x together with the set term r all x.

For fixed Γ and φ, a proof tree $\mathcal{T}$ in $\mathcal{A}(\mathcal{RC})$ is *confined* if every set term in $\mathcal{T}$ is either a subterm of a term in $\Gamma \cup \{\varphi\}$ or a subterm of the root of $\mathcal{T}$. (Note that this notion depends on Γ and on φ.) We write $\Gamma \vdash^{\text{confined}} \psi$ if $\Gamma \vdash \psi$ via a confined proof tree.

First, we show that what we have in Figure 3.4 deserves to be called an algorithm.

Remark Figure 3.4 describes an algorithm which, given a finite set $\Gamma \cup \{\varphi\}$ in $\mathcal{A}(\mathcal{RC})$, produces a model $\mathcal{M}$.

Here is the reason. For all set terms x and y, let us consider the question of whether or not $\Gamma \vdash^{\text{confined}} \mathsf{All}\ x\ y$. We can decide yes or no to this question by generating all proof trees which are built using the terms in Γ along with x and y, and all their subterms. In more detail, the set of consequences of Γ that interests us is the fixed point of a *monotone inductive definition* on a finite set.

Using this fact, here is how we build $\mathcal{M}$.

(i) Find the set M of subterms of terms in $\Gamma \cup \{\varphi\}$; there are only finitely many such terms;

(ii) Find the set W of all sentences $\mathsf{All}\ u\ v$ with the property that u and v either belong to M or are subterms of either x or y. This W is also finite.

(iii) Let $S_0 = \Gamma$

(iv) Given S_n, let S_{n+1} be the set of sentences in W which can be inferred in one step using the sentences in S_n and the rules in our logical system for $\mathcal{A}(\mathcal{RC})$.

(v) Since W is finite, there must be some n such that $S_n = S_{n+1}$.

Once we have W, we can read off the structure of the model $\mathcal{M}$.

Lemma 3.2.5 *For all $x \in M$, $[\![x]\!] = \{y \in M : \Gamma \vdash^{confined} \mathsf{All}\ y\ x\}$.*

Proof The proof is by induction on x. When x is a unary atom, our result is by the definition of $\mathcal{M}$. Assume our lemma holds for x; we prove that if r all x belongs to M, then

$$[\![r\ \mathsf{all}\ x]\!] \quad = \quad \{y \in M : \Gamma \vdash^{\text{confined}} \mathsf{All}\ y\ (r\ \mathsf{all}\ x)\}.$$

First, let $y \in [\![r\ \mathsf{all}\ x]\!]$. M is closed under subterms, by definition. Since r all x belongs to M, so also $x \in M$. Note that $\Gamma \vdash^{\text{confined}} \mathsf{All}\ x\ x$. Hence by induction hypothesis, $x \in [\![x]\!]$. So $(y, x) \in [\![r]\!]$. That is, $\Gamma \vdash^{\text{confined}} \mathsf{All}\ y\ (r\ \mathsf{all}\ x)$. In the other direction, assume that $\Gamma \vdash^{\text{confined}} \mathsf{All}\ y\ (r\ \mathsf{all}\ x)$. Suppose that $z \in [\![x]\!]$; we show that $(y, z) \in [\![r]\!]$. Then by induction hypothesis, $\Gamma \vdash^{\text{confined}} \mathsf{All}\ z\ x$. Now consider the following derivation tree:

$$\dfrac{\begin{matrix}\vdots \\ \mathsf{All}\ y\ (r\ \mathsf{all}\ x)\end{matrix} \qquad \dfrac{\begin{matrix}\vdots \\ \mathsf{All}\ z\ x\end{matrix}}{\mathsf{All}\ (r\ \mathsf{all}\ x)\ (r\ \mathsf{all}\ z)}}{\mathsf{All}\ y\ (r\ \mathsf{all}\ z)} \tag{3.3}$$

We claim that this tree is confined. Every term in the missing subtree on the left is either a subterm of $\Gamma \cup \{\varphi\}$ or of y or of r all x. Every term in the missing subtree on the right is either a subterm of $\Gamma \cup \{\varphi\}$ or of z or of x. But y and r all z are subterms of the root, z is a subterm of r all z, r all x belongs to M and is thus a subterm of $\Gamma \cup \{\varphi\}$, and x is a subterm of r all x. Thus every term in the derivation tree of (3.3) is either a subterm of $\Gamma \cup \{\varphi\}$ or is a subterm of the root of the tree. We conclude that $\Gamma \vdash^{\text{confined}} \mathsf{All}\ y\ (r\ \mathsf{all}\ z)$, and this goes to show that $(y, z) \in [\![r]\!]$. $\dashv$

Proposition 3.2.6 $\mathcal{M} \models \Gamma$.

This is Exercise 15, and it is an easy consequence of Lemma 3.2.5.

Theorem 3.2.7 *The algorithm in Figure 3.4 is correct. That is, if $\Gamma \not\vdash \varphi$, then $\mathcal{M}$ is a model of Γ where φ fails.*

Proof Recall that we fixed Γ and φ and then constructed $\mathcal{M}$ from them. By Proposition 3.2.6, $\mathcal{M} \models \Gamma$. We prove the contrapositive of our statement: if $\mathcal{M} \models \varphi$, then $\Gamma \vdash \varphi$. Suppose that $\mathcal{M} \models \varphi$, and write φ as $\mathsf{All}\ u\ v$. Then u and v belong to $M_0 \subseteq M$. By Lemma 3.2.5, $u \in [\![u]\!]$. So $u \in [\![v]\!]$. But then by the lemma again, $\Gamma \vdash^{\text{confined}} \mathsf{All}\ u\ v$. So $\Gamma \vdash \mathsf{All}\ u\ v$, as desired. $\dashv$

Corollary 3.2.8 *If $\Gamma \vdash \varphi$, then there is a derivation of φ from Γ all of whose set terms belong to $\Gamma \cup \{\varphi\}$.*

Proof Assume that $\Gamma \vdash \varphi$. Recall that the notion of *confined* used in Theorem 3.2.7 is defined from Γ and φ. We need only show that $\Gamma \vdash^{\text{confined}} \varphi$. By soundness, $\Gamma \models \varphi$. As mentioned in the proof of Theorem 3.2.7, the model $\mathcal{M}$ satisfies Γ. By soundness, $\mathcal{M} \models \varphi$. And then the argument in that proof shows exactly that $\Gamma \vdash^{\text{confined}} \varphi$. $\dashv$

Example 3.4 Here is a question: does the conclusion follow?

$$\begin{array}{l} \textsf{All boys men} \\ \textsf{All (see all women) (see all (see all boys))} \\ \textsf{All (see all (see all boys)) women} \\ \underline{\textsf{All women (see all men)}} \\ \textsf{All (see all boys) (see all men)} \end{array}$$

To answer this question, we take Γ to be the assumptions above, and M_0 then consists of the set terms

boys	(1)	see all women	(5)
see all boys	(2)	men	(6)
see all (see all boys)	(3)	women	(7)
see all men	(4)		

This set is closed under subterms. And so $M = M_0$. M^* would add see all (see all women), see all (see all men), and see all (see all (see all boys)).

It would take a fair amount of work to get the structure of the model by hand, but the algorithm is straightforward. We would get the following structure: for the unary atoms:

$$\begin{array}{lcl} [\![\textsf{boys}]\!] & = & \{\textsf{boys}\} \\ [\![\textsf{men}]\!] & = & \{\textsf{boys}, \textsf{men}\} \\ [\![\textsf{women}]\!] & = & \{\textsf{women}, \textsf{see all (see all boys)}\} \end{array}$$

And for the binary atom see, the structure is as shown in Example 3.1. The numbering above shows the correspondence between the elements of M and the points shown in Example 3.1. By Proposition 3.2.6, $\mathcal{M} \models \Gamma$. In this example we are interested in All (see all boys) (see all men). This sentence is false in $\mathcal{M}$. This shows that $\Gamma \not\models$ All (see all boys) (see all men).

Incidentally, $\mathcal{M}$ is isomorphic to the one depicted in Figure 3.2. The numbers correspond to the listing of the set terms earlier in this example.

Many verbs Our syntax and semantics of $\mathcal{A}(\mathcal{RC})$ allows the set $\mathbf{R}$ of binary atoms to be arbitrary. Yet in our examples, we always took it to be a singleton set {see}. We did this to simplify matters: all the points which we wanted to make up until now are may be illustrated with examples containing just one verb.

There are a few additional interesting phenomena that happen with more than one verb. The first is that we might want to work with *background assumptions that are not expressible in our logic so far.* For example, consider the following purported inference:

$$\begin{array}{l} \textsf{All women are football players} \\ \underline{\textsf{All football fans love all football players}} \\ \textsf{All football fans like all women} \end{array} \tag{3.4}$$

The way in which we are doing things so far, there is no relation between like and love in our models. And so the inference above fails. Yet there is a sense in which a competent speaker would agree to the conclusion given the premises. And we cannot add a premise such as "loving involves liking" because this is not expressible in the language. For this reason, we expand or overall framework.

Background assumptions on verbs In addition to our assumptions Γ, we can adopt a set Δ of *background assumptions* of the form $r \sqsubseteq s$ for $r, s \in \mathbf{R}$. In this case above, we might have

$$\Delta \quad = \quad \{\textsf{loves} \sqsubseteq \textsf{likes}\}$$

Definition A model $\mathcal{M}$ *respects* Δ if whenever $r \sqsubseteq s$ belongs to Δ, then $[\![r]\!] \subseteq [\![s]\!]$ in $\mathcal{M}$.

In the proof theory, we define $\Gamma; \Delta \vdash \varphi$ by adding to the proof system in Figure 3.3 the following rules:

$$\frac{\textsf{All } x \ (r \textsf{ all } y)}{\textsf{All } x \ (s \textsf{ all } y)} \ \textsc{Background}$$

We have one rule for each inequality assertion $r \sqsubseteq s$ in Δ.

Turning back to (3.4), we take $\Delta = \{\textsf{like} \sqsubseteq \textsf{love}\}$. And then (3.4) might be formalized as

$$\frac{\dfrac{\textsf{All fans (love all players)}}{\textsf{All fans (like all players)}} \ \textsc{Background} \qquad \dfrac{\textsf{All women players}}{\textsf{All (like all players) (like all women)}} \ \textsc{Anti}}{\textsf{All fans (like all women)}} \ \textsc{Barbara}$$

The main result on this system is left to you in Exercise 18.

3.3 Exercises

Exercise 11. Fill in the details in the last paragraph of the proof of Theorem 3.1.1.

Exercise 12. Suppose we dropped the last rule in Figure 3.1 and instead used the following rule:

$$\frac{\textsf{All } x \wedge y \textsf{ are } z \wedge z \qquad \textsf{All } x \wedge y \textsf{ are } a \wedge a}{\textsf{All } x \wedge y \textsf{ are } z \wedge a}$$

This rule is a special case of the rule which was dropped. Let us write $\Gamma \vdash^* \varphi$ to mean that there is a derivation of φ from Γ in the second proof system. This problem addresses the issue of whether $\Gamma \vdash \varphi$ is the same as $\Gamma \vdash^* \varphi$. The matter boils down to whether the two rules are in fact equivalent in the presence of the other rules.

Here are two ways which you can show this:

(i) Check that the arguments in Section 3 go through for the relation $\Gamma \vdash^* \varphi$. Therefore, that system is complete. Then say why we must have

$$\textsf{All } x \wedge y \textsf{ are } z \wedge w, \textsf{All } x \wedge y \textsf{ are } a \wedge b \vdash^* \textsf{All } x \wedge y \textsf{ are } z \wedge a. \tag{3.5}$$

(ii) Show directly that (3.5) holds by giving a derivation.

Exercise 13. Re-read Example 3.3, and check that the picture shown there really is the canonical model of $\{\textsf{All } p \textsf{ are } q\}$. That is, show that no edges are missing from the picture.

Exercise 14. Find the canonical model of

$$\Gamma \quad = \quad \{\textsf{All } p \textsf{ are } q, \textsf{All } q \textsf{ (see all } p\textsf{)}\} \ .$$

Exercise 15. Re-read the proof of Theorem 3.2.7. We are given Γ and φ, and from these we construct a model $\mathcal{M}$ in Figure 3.4. Show that $\mathcal{M} \models \Gamma$. [This amounts to a careful look at the notion of *confined* proof trees.]

Exercise 16. Let $\Gamma = \{\textsf{All } x\ y, \textsf{All } y\ z\}$, and let φ be $\textsf{All (see all } z\textsf{) (see all } x\textsf{)}$. The derivation below shows that $\Gamma \vdash \varphi$:

$$\dfrac{\dfrac{\textsf{All } y\ z}{\textsf{All (see all } z\textsf{) (see all } y\textsf{)}} \quad \dfrac{\textsf{All } x\ y}{\textsf{All (see all } y\textsf{) (see all } x\textsf{)}}}{\textsf{All (see all } z\textsf{) (see all } x\textsf{)}}$$

(i) Why is the tree above *not* confined?

(ii) Show that $\Gamma \vdash^{\text{confined}} \varphi$.

Exercise 17. Suppose that we take the logical system from Figure 3.3, drop the (Anti) rule and instead add the following rule

$$\dfrac{\textsf{All } x \textsf{ (see all } z\textsf{)} \quad \textsf{All } y\ z}{\textsf{All } x \textsf{ (see all } y\textsf{)}}\ \textsc{Anti}'$$

(Example 3.2 shows that (Anti$'$) can be derived in $\mathcal{A}(\mathcal{RC})$.) Show that the resulting logical system is sound and complete. [For the completeness, there are two ways to go. First, you can check that the *completeness proof* that we gave would go through for the new system. Alternatively, you can show how to derive all instances of (Anti) in the new system.]

$$\frac{}{(x,y,x)} \qquad \frac{}{(x,y,y)} \qquad \frac{(x,y,u) \quad (x,y,v) \quad (u,v,z)}{(x,y,z)}$$

Figure 3.5: The logic of All x which are y are z, written here (x, y, z).

Exercise 18. This exercise concerns the addition of background assumptions on verbs which we saw on page 35. Prove the following completeness theorem: Let $\Gamma \cup \varphi$ be a set of sentences in $\mathcal{A}(\mathcal{RC})$, and let Δ be a set of background assumptions. Then the following are equivalent:

(i) Every model of Γ which respects the assumptions in Δ satisfies φ.

(ii) $\Gamma; \Delta \vdash \varphi$.

Another variation We conclude this section with a few exercises on a logic for sentences of the form

All x which are y are z.

To save space, we abbreviate this by (x, y, z). We take this sentence to be true in a given model $\mathcal{M}$ if $[\![x]\!] \cap [\![y]\!] \subseteq [\![z]\!]$. Note that All x are y is semantically equivalent to (x, x, y).

Exercise 19. Prove that the logic of All x which are y are z in Figure 3.5 is complete. [Hint: you can do this with a one-point model, just as in Exercise 3.]

Exercise 20. Let Γ be a set of (x, y, z) sentences.

(i) Show that if $\Gamma \models (x, y, z)$, then $\Gamma \models (y, x, z)$.

(ii) Show that if $\Gamma \vdash (x, y, z)$, then $\Gamma \vdash (y, x, z)$.

(iii) Suppose that we remove the axiom (x, y, y), and in its place take the symmetry rule

$$\frac{(y,x,z)}{(x,y,z)}$$

Show that the new system is complete.

Exercise 21. Let φ be a sentence of $\mathcal{A}$, say $\varphi =$ All x are y. Let $\varphi^* = (x, x, y)$. If Γ is a set of sentences of the first fragment, let $\Gamma^* = \{\varphi^* : \varphi \in \Gamma\}$. It is easy to check that if $\Gamma \vdash \varphi$, then $\Gamma^* \vdash \varphi^*$. Prove the converse. We say that the proof system for assertions (x, y, z) is a *conservative extension* of the system for All.

Exercise 22. Find a way to show that the logic of binary intersections $x \wedge y \leq z \wedge w$ is a conservative extension of the logic of assertions (x, y, z).

4 $\mathcal{S}$: the logic of All p are q, Some p are q, and No p are q

The language $\mathcal{S}$ contains the syllogistic sentences in All, Some, No.

4.1 Syntax and semantics

Starting with a set $\mathbf{P}$ of nouns, we take the sentences of $\mathcal{S}$ to be those of the form All p are q, Some p are q, and No p are q, where p and q are nouns. We think of this language (and its relative $\mathcal{S}^\dagger$) in connection with *syllogisms*, this is why we use the letter $\mathcal{S}$.

For the semantics, we use models $\mathcal{M}$ that consist of a set M with interpretations $[\![p]\!]$ of the nouns. Then we define

$$\begin{array}{lll} \mathcal{M} \models \mathsf{All}\ p\ \mathsf{are}\ q & \text{iff} & [\![p]\!] \subseteq [\![q]\!] \\ \mathcal{M} \models \mathsf{Some}\ p\ \mathsf{are}\ q & \text{iff} & [\![p]\!] \cap [\![q]\!] \neq \emptyset \\ \mathcal{M} \models \mathsf{No}\ p\ \mathsf{are}\ q & \text{iff} & [\![p]\!] \cap [\![q]\!] = \emptyset \end{array}$$

Example 4.1 Perhaps the first semantic fact to check in this fragment is the interaction of All and Some. For example,

$$\{\mathsf{All}\ p\ \mathsf{are}\ q, \mathsf{Some}\ p\ \mathsf{are}\ r\} \models \mathsf{Some}\ q\ \mathsf{are}\ r. \tag{4.1}$$

To see this, let $\mathcal{M}$ be a model and assume that the two hypotheses are true in $\mathcal{M}$. Then $[\![p]\!] \subseteq [\![q]\!]$, and also $[\![p]\!] \cap [\![r]\!] \neq \emptyset$. Let $x \in [\![p]\!] \cap [\![r]\!]$. Then $x \in [\![p]\!]$ and also $x \in [\![r]\!]$. Since $x \in [\![p]\!]$ and $[\![p]\!] \subseteq [\![q]\!]$, we also have $x \in [\![q]\!]$. So $x \in [\![q]\!] \cap [\![r]\!]$. As a result, $[\![q]\!] \cap [\![r]\!] \neq \emptyset$. This means that $\mathcal{M} \models$ Some q are r. This argument holds for all $\mathcal{M}$, and so we have established the semantic assertion stated in (4.1).

Example 4.2 Here is an example of a semantic fact which is expressible in $\mathcal{S}$:

$$\{\mathsf{All}\ p\ \mathsf{are}\ v, \mathsf{All}\ q\ \mathsf{are}\ w, \mathsf{No}\ v\ \mathsf{are}\ w\} \models \mathsf{No}\ p\ \mathsf{are}\ q.$$

To see this, fix a model $\mathcal{M}$ where the hypotheses all hold. That is, $[\![p]\!] \subseteq [\![v]\!]$, $[\![q]\!] \subseteq [\![w]\!]$, and $[\![v]\!] \cap [\![w]\!] = \emptyset$. By elementary reasoning involving sets, we see that $[\![p]\!] \cap [\![q]\!] = \emptyset$. This means that $\mathcal{M} \models$ No p are q. Since $\mathcal{M}$ was an arbitrary model of the hypotheses, we are done.

$$\frac{}{\textsf{All } p \textsf{ are } p} \qquad \frac{\textsf{All } p \textsf{ are } n \quad \textsf{All } n \textsf{ are } q}{\textsf{All } p \textsf{ are } q}$$

$$\frac{\textsf{Some } p \textsf{ are } q}{\textsf{Some } q \textsf{ are } p} \qquad \frac{\textsf{Some } p \textsf{ are } q}{\textsf{Some } p \textsf{ are } p}$$

$$\frac{\textsf{All } q \textsf{ are } n \quad \textsf{Some } p \textsf{ are } q}{\textsf{Some } p \textsf{ are } n} \qquad \frac{\textsf{All } p \textsf{ are } n \quad \textsf{No } n \textsf{ are } q}{\textsf{No } p \textsf{ are } q}$$

$$\frac{\textsf{No } p \textsf{ are } q}{\textsf{No } q \textsf{ are } p} \qquad \frac{\textsf{No } p \textsf{ are } p}{\textsf{No } p \textsf{ are } q}$$

$$\frac{\textsf{No } p \textsf{ are } p}{\textsf{All } p \textsf{ are } q} \qquad \frac{\textsf{Some } p \textsf{ are } q \quad \textsf{No } p \textsf{ are } q}{\varphi}\ \mathrm{X}$$

Figure 4.1: The rules of $\mathcal{S}$.

Example 4.3 For fixed p and q, there are no models which satisfy both of the sentences below:

$$\textsf{Some } p \textsf{ are } q \qquad \text{and} \qquad \textsf{No } p \textsf{ are } q.$$

For this reason,

$$\{\textsf{Some } p \textsf{ are } q, \textsf{No } p \textsf{ are } q\} \models \varphi \tag{4.2}$$

for *any* sentence φ. That is, if we write out what it means for (4.2) to hold, we see that it does hold *vacuously*.

Example 4.4 Let us check that

$$\{\textsf{Some } p \textsf{ are } q, \textsf{Some } q \textsf{ are } n, \textsf{All } q \textsf{ are } m\} \not\models \textsf{Some } p \textsf{ are } n$$

by building a model in which the hypotheses hold and the conclusion fails. Let φ be Some p are q, and let ψ be Some q are n. (That is, we take the points of the model to be sentences.) We take for a model $M = \{\varphi, \psi\}$ with $[\![p]\!] = \{\varphi\}$, $[\![q]\!] = \{\varphi, \psi\}$, $[\![n]\!] = \{\psi\}$, $[\![m]\!] = \{\varphi, \psi\}$. It is clear that $\mathcal{M}$ has the desired properties.

Incidentally, the reason that we take the points of the model in Example 4.4 to be sentences and not (say) numbers is to foreshadow a general construction. See Exercise **??**.

4.2 Proof theory

See Figure 4.1 for the rules of $\mathcal{S}$.

Example 4.5 The first important derivation in the logic:

$$\dfrac{\textsf{All } n \textsf{ are } q \qquad \dfrac{\dfrac{\textsf{All } n \textsf{ are } p \quad \textsf{Some } n \textsf{ are } n}{\textsf{Some } n \textsf{ are } p}}{\textsf{Some } p \textsf{ are } n}}{\textsf{Some } p \textsf{ are } q}$$

That is, if there is a n, and if all ns are ps and also qs, then some p is a q.

Example 4.6 Corresponding to Example 4.2, let us check that

$$\{\textsf{All } p \textsf{ are } v, \textsf{All } q \textsf{ are } w, \textsf{No } v \textsf{ are } w\} \vdash \textsf{No } p \textsf{ are } q.$$

$$\dfrac{\dfrac{\textsf{All } q \textsf{ are } w \qquad \dfrac{\dfrac{\textsf{All } p \textsf{ are } v \quad \textsf{No } v \textsf{ are } w}{\textsf{No } p \textsf{ are } w}}{\textsf{No } w \textsf{ are } p}}{\textsf{No } p \textsf{ are } q}}{\textsf{No } q \textsf{ are } p} \tag{4.3}$$

The proof system has a principle relating Some and No. We have seen the semantic reason behind this principle in Example 4.3, and we incorporate this fact into the proof system by taking the rule of *ex falso quodlibet* (also called *ex contradictione quodlibet*) to our system[1]. This is the rule (X) in Figure 4.1.

Definition A set Γ is *inconsistent* if $\Gamma \vdash \varphi$ for all φ. Otherwise, Γ is *consistent*.

Theorem 4.2.1 *The logic in Figure 4.1 is sound and complete for* $\mathcal{S}$.

4.3 The classical syllogistic forms

There are fifteen valid classical syllogistic forms. Figure 4.2 shows some of them.

[1]Please do not confuse this with *reductio ad absurdum*. See page 73 for more on this.

$$\frac{\textsf{All } p \textsf{ are } q \quad \textsf{All } r \textsf{ are } p}{\textsf{All } r \textsf{ are } q}\ \textsc{Barbara} \qquad \frac{\textsf{No } p \textsf{ are } q \quad \textsf{All } r \textsf{ are } p}{\textsf{No } r \textsf{ are } q}\ \textsc{Celarent}$$

$$\frac{\textsf{All } p \textsf{ are } q \quad \textsf{Some } r \textsf{ are } p}{\textsf{Some } r \textsf{ are } q}\ \textsc{Darii} \qquad \frac{\textsf{No } p \textsf{ are } q \quad \textsf{Some } r \textsf{ are } p}{\textsf{Some } r \textsf{ are } q'}\ \textsc{Ferio}$$

$$\frac{\textsf{No } p \textsf{ are } q \quad \textsf{All } r \textsf{ are } q}{\textsf{No } r \textsf{ are } p}\ \textsc{Cesare} \qquad \frac{\textsf{All } p \textsf{ are } q \quad \textsf{No } r \textsf{ are } q}{\textsf{No } r \textsf{ are } p}\ \textsc{Camestres}$$

$$\frac{\textsf{No } p \textsf{ are } q \quad \textsf{Some } r \textsf{ are } q}{\textsf{Some } r \textsf{ are } p'}\ \textsc{Festino} \qquad \frac{\textsf{All } p \textsf{ are } q \quad \textsf{Some } r \textsf{ are } q'}{\textsf{Some } r \textsf{ are } p'}\ \textsc{Baroco}$$

Figure 4.2: The valid Aristotelian syllogistic forms of the first and second figures, with their traditional names. Usually these syllogistic forms are not written with the notation p' but rather with the English word *not*.

4.4 $\mathcal{S}^\dagger$: syllogistic logic with full negation on nouns

We now study the language $\mathcal{S}^\dagger$ which is strictly bigger than $\mathcal{S}$ in that it has noun-level negation rather than sentence-level negation.

In the syntax, we again begin with a set $\mathbf{P}$ of *(unary) atoms*. We use p, q, ..., for atoms. The idea once again is that unary atoms represent plural common nouns. Let

$$\mathbf{Lit} \quad = \quad \mathbf{P} \cup \{p' : p \in \mathbf{P}\}.$$

In other words, we have two copies of $\mathbf{P}$, using the symbol $'$ to distinguish the copies. We call the elements of this set *literals* following uses in areas of logic. Once again, the elements of $\mathbf{Lit}$ as either atoms p, q, etc., or as *complemented atoms* p', q', ... Moreover, we extend this idea of complementation to a function *complementation operation* $'$: $\mathbf{Lit} \to \mathbf{Lit}$ on the literals such that $p'' = p$ for all literals p. (Yes, we use the same letters p, q, etc. to range over literals in this section.) This *involutive* property implies that complementation is a bijection on $\mathbf{Lit}$. Then we consider sentences All p are q and Some p are q. Here p and q are any literals, including the case when they are the same. We call this language $\mathcal{S}^\dagger$. We shall use letters like φ to denote sentences.

Semantics One starts with a set M and a subset $[\![p]\!] \subseteq M$ for each literal p, subject to the requirement that $[\![p']\!] = M \setminus [\![p]\!]$ for all p. This gives a *model* $\mathcal{M} = (M, [\![\ ]\!])$. We then define the satisfaction relation $\mathcal{M} \models \varphi$ just as in earlier sections, and also derived notions such as $\Gamma \models \varphi$.

$$\frac{}{\textsf{All } p \textsf{ are } p}\ \textsc{Axiom} \qquad \frac{\textsf{Some } p \textsf{ are } q}{\textsf{Some } p \textsf{ are } p}\ \textsc{Some}_1 \qquad \frac{\textsf{Some } p \textsf{ are } q}{\textsf{Some } q \textsf{ are } p}\ \textsc{Some}_2$$

$$\frac{\textsf{All } p \textsf{ are } n \quad \textsf{All } n \textsf{ are } q}{\textsf{All } p \textsf{ are } q}\ \textsc{Barbara} \qquad \frac{\textsf{All } q \textsf{ are } n \quad \textsf{Some } p \textsf{ are } q}{\textsf{Some } p \textsf{ are } n}\ \textsc{Darii}$$

$$\frac{\textsf{All } q \textsf{ are } q'}{\textsf{All } q \textsf{ are } p}\ \textsc{Zero} \qquad \frac{\textsf{All } q' \textsf{ are } q}{\textsf{All } p \textsf{ are } q}\ \textsc{One}$$

$$\frac{\textsf{All } q \textsf{ are } p'}{\textsf{All } p \textsf{ are } q'}\ \textsc{Antitone} \qquad \frac{\textsf{All } p \textsf{ are } q \quad \textsf{Some } p \textsf{ are } q'}{S}\ \textsc{X}$$

Figure 4.3: Rules for $\mathcal{S}^\dagger$

Example 4.7 We claim that $\Gamma \models \textsf{All } x \textsf{ are } z$, where

$$\Gamma \quad = \quad \{\textsf{All } y' \textsf{ are } p, \textsf{All } p \textsf{ are } q, \textsf{All } q \textsf{ are } y, \textsf{All } y \textsf{ are } p, \textsf{All } q \textsf{ are } z\}.$$

Here is an informal explanation. Since all y and all y' are p, everything whatsoever is a p. And since all p are q, and all q are y, we see that everything is a y. In particular, all x are y. But the last two premises and the fact that all p are q also imply that all y are z. So all x are z.

Exercise 23. Here is an example which we mention mostly as a challenge. Let Γ be the set of the sentences below:

$$\textsf{All } y \textsf{ are } x, \textsf{All } y' \textsf{ are } x, \textsf{All } z' \textsf{ are } y, \textsf{All } z \textsf{ are } y', \textsf{All } z \textsf{ are } w.$$

It is *not* true that

$$\Gamma \vdash \textsf{All } y \textsf{ are } w.$$

Find a model $\mathcal{M} \models \Gamma$ where $[\![y]\!] \not\subseteq [\![w]\!]$. The point of this exercise is that the details of the completeness proof for our logic will give us a way of *automatically* solving problems like this!

We shall see the solution later.

No In previous work, we took No p are q as a basic sentence in the syntax. There is no need to do this here: we may regard No p are q as a variant notation for All p are q'. In other words, if one wants to add No as a basic sentence forming-operation, on a par with Some and All, it would be easy to do so.

Proof trees We have discussed the meager syntax of $\mathcal{S}^{\dagger}$ and its semantics. We next turn to the proof theory whose rules are listed in Figure 4.3.

We attached names to the rules in Figure 4.3 so that we can refer to them later. We usually do not display the names of rules in our proof trees except when to emphasize some point or other. The names "Barbara" and "Darii" are traditional from Aristotelian syllogisms. But the (*Antitone*) rule is not part of traditional syllogistic reasoning. It is possible to drop (*Some*$_2$) if one changes the conclusion of (*Darii*) to Some n are p. But at one point it will be convenient to have (*Some*$_2$), and so this guides the formulation. The rules (*Zero*) and (*One*) are concerned with what is often called vacuous universal quantification. That is, if $q' \subseteq p$, then q is the whole universe and q' is empty; so q is a superset of every set and q' a subset. We have already seen the rule (X) rule: it permits inference of any sentence φ whatsoever from a contradiction.

Example 4.8 Returning to Example 4.7, here is a proof tree showing $\Gamma \vdash$ All x are z:

$$\dfrac{\dfrac{\dfrac{\mathsf{All}\ y'\ \mathsf{are}\ p \quad \dfrac{\mathsf{All}\ p\ \mathsf{are}\ q \quad \mathsf{All}\ q\ \mathsf{are}\ y}{\mathsf{All}\ p\ \mathsf{are}\ y}}{\mathsf{All}\ y'\ \mathsf{are}\ y}}{\mathsf{All}\ x\ \mathsf{are}\ y} \qquad \dfrac{\mathsf{All}\ y\ \mathsf{are}\ p \quad \dfrac{\mathsf{All}\ p\ \mathsf{are}\ q \quad \mathsf{All}\ q\ \mathsf{are}\ z}{\mathsf{All}\ p\ \mathsf{are}\ z}}{\mathsf{All}\ y\ \mathsf{are}\ z}}{\mathsf{All}\ x\ \mathsf{are}\ z}$$

4.5 Orthoposets and their representations

An important step in our work is to develop an *algebraic semantics* for $\mathcal{S}^{\dagger}$. There are several definitions, and then a *representation theorem.* As with other uses of algebra in logic, the point is that the representation theorem is also a *model construction technique.*

An *orthoposet* is a tuple $\mathbb{P} = (P, \leq, 0, ')$ such that

(i) $(P, \leq)$ is a partial order: $\leq$ is a reflexive, transitive, and antisymmetric relation on the set P.

(ii) 0 is a minimum element: $0 \leq p$ for all $p \in P$.

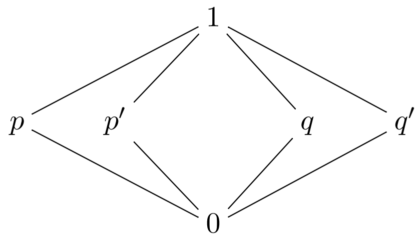
(iii) $x \mapsto x'$ is an antitone map in both directions: $x \leq y$ iff $y' \leq x'$.

(iv) $x \mapsto x'$ is involutive: $x'' = x$.

(v) complement inconsistency: If $x \leq y$ and $x \leq y'$, then $x = 0$.

The notion of an orthoposet mainly appears in papers on quantum logic. (In fact, the stronger notion of an *orthomodular poset* appears to be more central there. However, I do not see any application of this notion to logics like $\mathcal{S}^\dagger$.)

Example 4.9 The example below is sometimes called the *Chinese lantern*, and we'll call it $\mathbb{P}_2$.[2]



Here and elsewhere, we understand $(x')' = x$, $0' = 1$, $1' = 0$.

Example 4.10 For example, for all sets p we have an orthoposet $(\mathcal{P}(X), \subseteq, \emptyset, ')$, where $\subseteq$ is the inclusion relation, $\emptyset$ is the empty set, and $a' = X \setminus a$ for all subsets a of p.

Definition Let Γ be any set of sentences in $\mathcal{S}^\dagger$. Γ need not be consistent. Definition 2.4 defines the fundamental relation $\leq$ from Γ and the logic, and Proposition 2.4.1 shows this relation to be a preorder. We have an induced equivalence relation $\equiv$, and we take $\mathbf{Lit}_\Gamma$ to be the quotient set $\mathbf{Lit}/\!\equiv$. That is, we define $u \equiv v$ to mean $u \leq v$ and $v \leq u$. This quotient set $\mathbf{Lit}/\!\equiv$ is a poset under the induced relation: if $[u] \leq [v]$ and $[v] \leq [u]$, then $u \equiv v$ so that $[u] = [v]$. If there is some p such that $p \leq p'$, then for all q we have $[p] \leq [q]$ in $\mathbf{Lit}/\!\equiv$ In this case, set 0 to be $[p]$ for any such p. (If such p exists, its equivalence class is unique.) We finally define $[p]' = [p']$. If there is no p such that $p \leq p'$, we add fresh elements 0 and 1 to $\mathbf{Lit}/\!\equiv$. We then stipulate that $0' = 1$, and that for all $x \in \mathbb{P}_\Gamma$, $0 \leq x \leq 1$.

It is not hard to check that we have an orthoposet $\mathbf{Lit}/\!\equiv \; = (\mathbf{Lit}/\!\equiv, \leq, 0, ')$. The antitone property comes from the axiom with the same name, and the complement inconsistency is verified using the similarly-named part of Lemma 26.

Example 4.11 This example pertains to Example 23 from before. Let Γ be defined by

$$\Gamma = \{\mathsf{All}\ y\ \mathsf{are}\ x, \mathsf{All}\ y'\ \mathsf{are}\ x, \mathsf{All}\ z'\ \mathsf{are}\ y, \mathsf{All}\ z\ \mathsf{are}\ y', \mathsf{All}\ z\ \mathsf{are}\ w\}.$$

[2]The more standard name for this seems to be $M02$, with MO standing for *modular ortholattice*.

Then

$$\begin{array}{ll} [x] = \{x\} & [x'] = \{x'\} \\ [y] = \{y, z'\} & [y'] = \{y', z\} \\ [z] = \{y', z\} & [z'] = \{y, z'\} \\ [w] = \{w\} & [w'] = \{w'\} \end{array}$$

Here is a picture of the orthoposet $\mathbb{P}_\Gamma$:

$$\begin{tikzcd}
 & {[x]} \arrow[dl, dash] \arrow[dr, dash] & \\
{[z'] = [y]} \arrow[d, dash] & & {[w]} \arrow[d, dash] \\
{[w']} \arrow[dr, dash] & & {[y'] = [x]} \arrow[dl, dash] \\
 & {[x']} &
\end{tikzcd}$$

Definition Let $\mathbb{P}$ and $\mathbb{Q}$ be orthoposets. A *morphism of orthoposets* $f : \mathbb{P} \to \mathbb{Q}$ is a is a map $m : P \to Q$ preserving the order (if $x \leq y$, then $mx \leq my$), the complement $m(x') = (mx)'$, and minimum elements ($m0 = 0$). We say m is *strict* if the following extra condition holds: $x \leq y$ iff $mx \leq my$.

Definition A *point* of an orthoposet $\mathbb{P} = (P, \leq, 0, ')$ is a subset $S \subseteq P$ with the following properties:

(i) If $p \in S$ and $p \leq q$, then $q \in S$ (S is *up-closed*).

(ii) For all p, either $p \in S$ or $p' \in S$ (S is *complete*), but not both (S is *consistent*).

Example 4.12 Let $X = \{1, 2, 3\}$, and let $\mathcal{P}(X)$ be the power set orthoposet from Example 4.10. Then S is a point, where

$$S = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

(More generally, if p is any finite set, then the collection of subsets of p containing more than half of the elements of p is a point of $\mathcal{P}(X)$.) Also, it is easy to check that the points on this $\mathcal{P}(X)$ are exactly S as above and the three sets T_1, U_2, and T_3, where

$$T_i = \{A \subseteq X : i \in A\}.$$

If you know about *ultrafilters* of boolean algebras, then you should compare the definition of a *point* with that of an ultrafilter. (And if you do not know about ultrafilters, please skip over this paragraph.) Both definitions have something in common and both serve the same purpose in being related to a representation theorem. But there is a formal difference as well: the point S in Example 4.12 shows that a point of a boolean algebra need not be an ultrafilter or even a filter.

Lemma 4.5.1 *For a subset S_0 of an orthoposet $\mathbb{P} = (P, \leq, ')$, the following are equivalent:*

(i) S_0 is a subset of a point S of $\mathbb{P}$.

(ii) For all $x, y \in S_0$, $x \not\leq y'$.

Proof Clearly (1) $\Longrightarrow$ (2). For the more important direction, use Zorn's Lemma to get a $\subseteq$-maximal superset S_1 of S_0 with the consistency property. Let $S = \{q : (\exists p \in S_1) q \geq p\}$. So S is up-closed. We check that consistency is not lost: suppose that $r, r' \in S$. Then there are $q_1, q_2 \in S_1$ such that $r \geq q_1$ and $r' \geq q_2$. But then $q_2' \geq r \geq q_1$. Since $q_1 \in S_1$, so too $q_2' \in S_1$. Thus we see that S_1 is not consistent, and this is a contradiction. To conclude, we only need to see that for all $r \in P$, either r or r' belongs to S. If $r \notin S$, then $r \notin S_1$. By maximality, there is $q \in S_1$ such that $q_1 \leq r'$. (For otherwise, $S_1 \cup \{r\}$ would be a consistent proper superset of S_1.) And as $r' \notin S$, there is $q_2 \in S_1$ such that $q_2 \leq r$. Then as above $q_1 \leq q_2'$, leading to the same contradiction. ⊣

We now present a representation theorem that implies the completeness of the logic. It is due to Calude, Hertling, and Svozil [**?**]. We also state an additional technical point.

Theorem 4.5.2 (Representation Theorem for Orthoposets [?, ?, ?]) *Let $\mathbb{P} = (P, \leq, ')$ be an orthoposet. There is a set* points(P) *and a strict morphism of orthoposets $m : \mathbb{P} \to \mathcal{P}(\text{points}(P))$.*

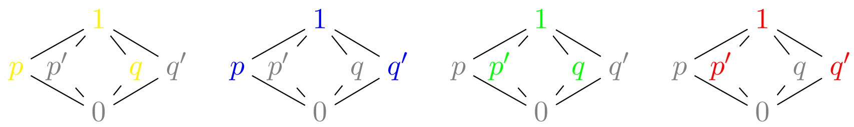
Moreover, if $S \cup \{p\} \subseteq P$ has the following two properties, then $m(p) \setminus \bigcup_{q \in S} m(q)$ is non-empty:

(i) For all $q \in S$, $p \not\leq q$.

(ii) For all $q, r \in S$, $q \not\geq r'$.

Proof Let *points*(P) be the collection of points of $\mathbb{P}$. The map m is defined by $m(p) = \{S : p \in S\}$. The preservation of complement comes from the completeness and consistency requirement on points, and the preservation of order from the up-closed-ness. Clearly $m0 = \emptyset$. We must check that if $q \not\geq p$, then there is some point S such that $p \in S$ and $q \notin S$. For this, take $S = \{q\}$ in the "moreover" part. And for that, let $T = \{p\} \cup \{q' : q \in S\}$. Lemma 4.5.1 applies, and so there is some point $u \supseteq T$. Such u belongs to $m(p)$. But if $q \in S$, then $q' \in T \subseteq U$; so u does not belong to $m(q)$. ⊣

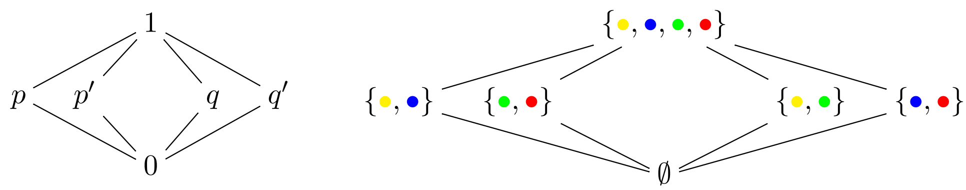
Example 4.13 Here is a picture which illustrates the workings of the Representation Theorem 4.5.2. We presented an orthoposet $\mathbb{P}_2$ in Example 4.9. Three subsets of the underlying set are points, and these are shown in colors below[3].



We call these points ${\color{yellow}\bullet}$, ${\color{blue}\bullet}$, ${\color{green}\bullet}$, and ${\color{red}\bullet}$. The orthoposet

$$\mathbb{Q} \quad = \quad \mathcal{P}(\{{\color{yellow}\bullet}, {\color{blue}\bullet}, {\color{green}\bullet}, {\color{red}\bullet}\})$$

has sixteen elements, so we shall not display it. But we can illustrate the function $m : \mathbb{P}_2 \to \mathbb{Q}$. For example, $m(p) = \{{\color{yellow}\bullet}, {\color{blue}\bullet}\}$, because the points to which p belongs are ${\color{yellow}\bullet}$ and ${\color{blue}\bullet}$. Similarly, $m(0) = \emptyset$. Here is a picture of $\mathbb{P}_2$ and its image under m inside $\mathbb{Q}$:



4.6 Completeness

The completeness theorem is based on algebraic machinery that we have just seen.

Lemma 4.6.1 (Pratt-Hartmann) *Suppose that* $\Gamma \models$ Some p are q. *Then there is some sentence in* Γ_{some}, *say* Some a are b, *such that*

$$\Gamma_{all} \cup \{\text{Some } a \text{ are } b\} \models \text{Some } p \text{ are } q.$$

Proof If not, then for every $\varphi \in \Gamma_{some}$, there is a model $\mathcal{M}_\varphi \models \Gamma_{all} \cup \{\varphi\}$ and $[\![p]\!] \cap [\![q]\!] = \emptyset$ in the model. Take the disjoint union of the models $\mathcal{M}_\varphi$ to get a model of $\Gamma_{all} \cup \Gamma_{some} = \Gamma$ where Some p are q fails. $\dashv$

Lemma 4.6.2 *Let* $\Gamma \subseteq \mathcal{S}^\dagger$. *There is a model* $\mathcal{M} = (M, [\![\ ]\!])$ *such that*

(*i*) $\mathcal{M} \models \Gamma_{all}$.

[3] If you cannot see the colors, the four points are $\{p, q, 1\}$, $\{p', q, 1\}$, $\{p, q', 1\}$, and $\{p', q', 1\}$.

(ii) If $\varphi \in \mathcal{A}$ and $\mathcal{M} \models \varphi$, then $\Gamma \vdash \varphi$.

(iii) If Γ is consistent, then also $\mathcal{M} \models \Gamma_{some}$.

Proof We write $\mathbb{P}_\Gamma$ for the orthoposet $\mathbf{Lit}/\!\equiv$. (Recall that this was obtained by from the set of literals by taking quotient by $p \equiv q$ iff $p \leq q \leq p$. See Definition 4.5.) Let n be the natural map of $\mathbf{P}$ into $\mathbb{P}_\Gamma$, taking an atom p to its equivalence class $[p]$. Even though $\mathbf{P}$ is not an orthoposet with its order $\leq$, it *is* a preorder (see Proposition 2.4.1). This map n is an order-preserving map from one preorder $\mathbf{P}$ into another. Moreover, n preserves the order in both directions. We also apply Theorem 4.5.2, to obtain a strict morphism of orthoposets m as shown below:

$$\mathbf{P} \xrightarrow{\ n\ } \mathbb{P}_\Gamma \xrightarrow{\ m\ } \mathit{points}(\mathbb{P}_\Gamma)$$

Let $M = \mathit{points}(\mathbb{P}_\Gamma)$, and let $[\![\ ]\!] : \mathbb{P}_\Gamma \to \mathcal{P}(M)$ be the composition $m \circ n$, regarded as an order-preserving function on preorders. We thus have a model $\mathcal{M} = (\mathit{points}(\mathbb{P}_\Gamma), [\![\ ]\!])$.

We check that $\mathcal{M} \models \Gamma$. Note that n and m are strict monotone functions. So the semantics has the property that the All sentences holding in $\mathcal{M}$ are exactly the consequences of Γ. We turn to a sentence in Γ_{some} such as Some u are v. Assuming the consistency of Γ, $u \not\leq v'$. Thus $[\![u]\!] \not\subseteq ([\![v]\!])'$. That is, $[\![u]\!] \cap [\![v]\!] \neq \emptyset$. $\dashv$

Unfortunately, the last step in this proof is not reversible, in the following precise sense. It does not follow from $u \not\leq v'$ that $\Gamma \vdash$ Some u are v. (For example, if Γ is the empty set we have $u \not\leq v'$, and indeed $\mathcal{M}(\Gamma) \models$ Some u are v. But Γ only derives valid sentences.)

Theorem 4.6.3 (Completeness of the proof system for $\mathcal{S}^\dagger$) *$\Gamma \vdash \varphi$ iff $\Gamma \models \varphi$.*

Proof As always, the soundness half is trivial. Suppose that $\Gamma \models \varphi$; we show that $\Gamma \vdash \varphi$. We may assume that Γ is consistent.

If φ is a sentence All p are q, consider $\mathcal{M}(\Gamma)$ from Lemma 4.6.2. It is a model of Γ, hence of φ; and then by the property the second part of the lemma, $\Gamma \vdash \varphi$.

For the rest of this proof, let φ be Some p are q. From Γ and φ, we find a and b satisfying the conclusion of Lemma 4.6.1.

We again use Lemma 4.6.2 and consider the model $\mathcal{M} = \mathcal{M}(\mathbf{Lit}/\!\equiv_{\Gamma_{all}})$ of points on $\mathbf{Lit}/\!\equiv_{\Gamma_{all}}$. $\mathcal{M} \models \Gamma_{all}$.

Consider $\{[a], [b], [p']\}$. If this set were a subset of a point x, then consider $\{x\}$ as a one-point submodel of $\mathcal{M}$. In the submodel, $\Gamma_{all} \cup \{$Some a are $b\}$ would hold, and yet Some p are q would fail since $[\![p]\!] = \emptyset$.

We use Lemma 4.5.1 to divide into cases:

(i) $a \leq a'$.

(ii) $a \leq b'$.

(iii) $a \leq p$.

(iv) $b \leq b'$.

(v) $b \leq p$.

(vi) $p' \leq p$.

(More precisely, the first case would be $[a] \leq [a']$. By strictness of the natural map, this means that $a \leq a'$; that is, $\Gamma_{all} \vdash$ All a are a'.) In cases (1), (2), and (4), we easily see that Γ is inconsistent, contrary to the assumption at the outset. Case (6) implies that both (3) and (5) hold. Thus we may restrict attention to (3) and (5).

Next, consider $\{a, b, q'\}$. The same analysis gives two other cases, independently: $a \leq q$, and $b \leq q$. Putting these together with the other two gives four pairs. The following are representative:

$\underline{a \leq p \text{ and } b \leq q}$: Using Some a are b, we see that $\Gamma \vdash$ Some p are q.

$\underline{a \leq p \text{ and } a \leq q}$: We first derive Some a are b, and then again we see $\Gamma \vdash$ Some p are q.

This completes the proof. ⊣

Recall that *consistency* is a syntactic concept; a set Γ is consistent if $\Gamma \not\vdash$ Some p are p'. The matching semantic concept is *satisfiability*: Γ is satisfiable iff it has a model.

Corollary 4.6.4 *A set* $\Gamma \subseteq \mathcal{S}^\dagger$ *is consistent iff it is satisfiable.*

4.7 Algorithmic analysis

At this point we have the completeness of the logical system for $\mathcal{S}^\dagger$. We'd like to go a bit further and give an efficient algorithm to tell whether, given a finite $\Gamma \subseteq \mathcal{S}^\dagger$ and some $\varphi \in \mathcal{S}^\dagger$, $\Gamma \vdash \varphi$ or not. Further, if $\Gamma \not\vdash \varphi$, we'd like an algorithm to construct a counterexample. Our leading ideas come from work we did on $\mathcal{A}$ in Section 2, especially the graph G_Γ and Theorem 2.5.1.

Fix a finite set $\Gamma \subseteq \mathcal{S}^\dagger$. We construct a graph G_Γ (different from the one in Section 2) in several steps.

(a) The points of G_Γ are the elements of **Lit** which occur in Γ.

(b) First, we put $p \underset{\Gamma}{\rightarrow} q$ if Γ contains either All p are q or All q' are p'.

(c) Second, we define $p \overset{*}{\underset{\Gamma}{\rightarrow}} q$ if there is a path of length at least zero[4] from p to q following $\underset{\Gamma}{\rightarrow}$.

(d) Third, we say that a point p is a *zero-point* if $p \overset{*}{\underset{\Gamma}{\rightarrow}} p'$, and p is a *one-point* if $p' \overset{*}{\underset{\Gamma}{\rightarrow}} p$. (A point could be both a zero-point and a one-point. Also p is a zero-point iff p' is a one-point.)

(e) Finally, we write $p \overset{**}{\underset{\Gamma}{\rightarrow}} q$ if either $p \overset{*}{\underset{\Gamma}{\rightarrow}} q$, or if p is a zero-point, or if q is a one-point.

Lemma 4.7.1 *Concerning* $\overset{*}{\underset{\Gamma}{\rightarrow}}$ *and* $\overset{**}{\underset{\Gamma}{\rightarrow}}$ *:*

[4]There always is a path of length zero from a point in a graph to itself. So for all points p in G_Γ, $p \overset{*}{\underset{\Gamma}{\rightarrow}} p$.

(i) The relation $\overset{**}{\underset{\Gamma}{\Rightarrow}}$ *is transitive.*

(ii) If $p \overset{*}{\underset{\Gamma}{\Rightarrow}} q$, *then* $q' \overset{*}{\underset{\Gamma}{\Rightarrow}} p'$.

(iii) If $p \overset{**}{\underset{\Gamma}{\Rightarrow}} q$, *then* $q' \overset{**}{\underset{\Gamma}{\Rightarrow}} p'$.

(iv) If $p \overset{**}{\underset{\Gamma}{\Rightarrow}} p'$, *then* $p \overset{*}{\underset{\Gamma}{\Rightarrow}} p'$.

Proof For (1), assume that $p \overset{**}{\underset{\Gamma}{\Rightarrow}} q$ and $q \overset{**}{\underset{\Gamma}{\Rightarrow}} r$. Then there are nine cases, and in each of these, we see that $p \overset{**}{\underset{\Gamma}{\Rightarrow}} r$. Part (2) is an easy induction on the lengths of paths showing $p \overset{*}{\underset{\Gamma}{\Rightarrow}} q$. Part (3) is a direct verification, using the fact that p is a zero-point iff p' is a one-point. Part (4) also follows easily from the definitions. ⊣

Theorem 4.7.2 *Let* $\Gamma \subseteq \mathcal{S}^{\dagger}$. *The following are equivalent:*

(i) $\Gamma \vdash$ All p are q *in our system for* $\mathcal{S}^{\dagger}$.

(ii) $p \overset{**}{\underset{\Gamma}{\Rightarrow}} q$.

Proof For (1)$\Longrightarrow$(2), we argue by induction on proofs in our system for $\mathcal{S}^{\dagger}$ that if $\Gamma \vdash$ All p are q, then $p \overset{**}{\underset{\Gamma}{\Rightarrow}} q$. Note first that a proof from Γ must only use the sentences of Γ, the axioms All p are p, and the rules (Barbara), (Antitone), (Zero), and (One). The base case is for a proof tree consisting of a sentence in Γ or an axiom. In the first of these cases, we put $p \underset{\Gamma}{\Rightarrow} q$ by point (b). In the second, we use (c) with a path of length zero. For the induction step, we break into cases as to the rule at the root, and in all cases we use Lemma 4.7.1. For example, if the rule at the root is (Barbara), deriving All p are q from All p are r and All r are q, then by induction hypothesis $p \overset{**}{\underset{\Gamma}{\Rightarrow}} r$ and $r \overset{**}{\underset{\Gamma}{\Rightarrow}} q$. By the first part of the lemma, $p \overset{**}{\underset{\Gamma}{\Rightarrow}} q$. The induction step for (Antitone) similarly uses the third part of the lemma. Here is the induction step for (Zero); the step for (One) is similar. Suppose $\Gamma \vdash$ All p are q using a tree whose last step applies (Zero) to All p are p'. By induction hypothesis, $p \overset{**}{\underset{\Gamma}{\Rightarrow}} p'$. By part (4) of the lemma, $p \overset{*}{\underset{\Gamma}{\Rightarrow}} q$, so p is a zero-point. Then we have $p \overset{**}{\underset{\Gamma}{\Rightarrow}} q$, as desired.

In the other direction, we show that if $p \overset{**}{\underset{\Gamma}{\Rightarrow}} q$, then $\Gamma \vdash$ All p are q. If $p \underset{\Gamma}{\Rightarrow} q$, then clearly $\Gamma \vdash$ All p are q with a one-point tree, or perhaps a two-point tree. An easy induction on the number n shows that if there is a path in $\underset{\Gamma}{\Rightarrow}$ of length n from p to q, then again $\Gamma \vdash$ All p are q. This shows that if $p \overset{*}{\underset{\Gamma}{\Rightarrow}} q$, then $\Gamma \vdash$ All p are q. So if p is a zero-point, then $\Gamma \vdash$ All p are p. This means that for all q, $\Gamma \vdash$ All p are q. Similarly, if q is a one-point, then for all p, $\Gamma \vdash$ All p are q. From these last two observations, if $p \overset{**}{\underset{\Gamma}{\Rightarrow}} q$, $\Gamma \vdash$ All p are q. ⊣

Running time If G_Γ has n points (that is, if there are n literals in Γ), then the relation $\overset{**}{\underset{\Gamma}{\Rightarrow}}$ may be computed in time $O(n^3)$. This is a standard result in algorithms.

An algorithm to tell if Γ is consistent or not At this point, we give an algorithm to solve the first main problem about $\mathcal{S}^\dagger$, to tell whether a given finite set Γ is consistent. Our work is based on the following result.

Lemma 4.7.3 Γ *is inconsistent iff there is a sentence* Some p are q *in* Γ_{some} *such that* $\Gamma_{all} \vdash$ All p are q'.

We leave the proof to you as Exercise 28. Based on this fact, here is how we can tell if Γ is consistent. First, compute the relation $\overset{**}{\underset{\Gamma}{\Rightarrow}}$. Then go through the sentences Some p are q in Γ_{some}, and for each of them, see if $p \overset{**}{\underset{\Gamma}{\Rightarrow}} q$ or not. By Theorem 4.7.2; there is such a sentence iff Γ is inconsistent.

An algorithm to tell if $\Gamma \vdash$ All p are q or not To tell if $\Gamma \vdash$ All p are q in this logic, we first check if Γ is consistent. If not, then $\Gamma \vdash$ All p are q. (Why is this?) We thus assume that Γ is consistent. We claim that $\Gamma \vdash$ All p are q iff $\Gamma_{all} \vdash$ All p are q. (By induction on proofs in our current system, any proof in this system whose root is an All sentence and which does not use (X) must consist entirely of All sentences.) We build G_Γ, and compute the relation $\overset{**}{\underset{\Gamma}{\Rightarrow}}$. Then see if $p \overset{**}{\underset{\Gamma}{\Rightarrow}} q$ or not.

An algorithm to tell if $\Gamma \vdash$ Some p are q or not We need a bit of preliminary work. As before, we may assume that Γ is consistent. By Lemma 4.6.1 and Theorem 4.6.3, there is a sentence Some a are b in Γ_{some} such that

$$\Gamma_{all} \cup \{\text{Some } a \text{ are } b\} \vdash \text{Some } p \text{ are } q$$

The proof may not use (X), since Γ is consistent. Define $\overset{**}{\underset{\Gamma}{\Rightarrow}}$ from Γ_{all}. By Exercise 29, we have $a \overset{**}{\underset{\Gamma}{\Rightarrow}} p$ or $b \overset{**}{\underset{\Gamma}{\Rightarrow}} p$, and also that $a \overset{**}{\underset{\Gamma}{\Rightarrow}} q$ or $b \overset{**}{\underset{\Gamma}{\Rightarrow}} q$.

Thus to see whether a consistent Γ derives Some p are q, look for a sentence Some a are b in Γ such that either $a \overset{**}{\underset{\Gamma}{\Rightarrow}} p$ or $b \overset{**}{\underset{\Gamma}{\Rightarrow}} p$, and also either $a \overset{**}{\underset{\Gamma}{\Rightarrow}} q$ or $b \overset{**}{\underset{\Gamma}{\Rightarrow}} q$. If such a sentence Some a are b in Γ exists, then $\Gamma \vdash$ All p are q, and vice-versa.

An algorithm to build a model of a consistent set Γ If Γ is consistent, then the model $\mathcal{M}$ from Lemma 4.6.2 satisfies Γ. Suppose that Γ has n literals. The universe M of this model is the set of points on an orthoposet of size at most n, and so M has size at most 2^n. The rest of the construction of M is polynomial.

4.8 Exercises

In these exercises, $\Gamma \vdash \varphi$ refers to derivability in our current system.

Exercise 24. Show that

$$\text{All } y \text{ are } p, \text{All } y' \text{ are } p \vdash \text{All } x \text{ are } p.$$

Exercise 25. Show that

$$\textsf{All } y \textsf{ are } p, \textsf{All } y' \textsf{ are } p, \textsf{All } q \textsf{ are } z, \textsf{Some } x \textsf{ are } z' \vdash \textsf{Some } p \textsf{ are } q'.$$

Exercise 26. Show the following:

(i) Some p are $p' \vdash \varphi$ (a contradiction fact)

(ii) All p are n, No n are $q \vdash$ No q are p (Celarent)

(iii) No p are $q \vdash$ No q are p (E-conversion)

(iv) Some p are q, No q are $n \vdash$ Some p are n' (Ferio)

(v) All q are n, All q are $n' \vdash$ No q are q (complement inconsistency)

Exercise 27. Prove Corollary 4.6.4, using Theorem 4.6.3.

Exercise 28. Γ is inconsistent iff there is a sentence Some p are q in Γ_{some} such that $\Gamma_{all} \vdash$ All p are q'.

[This is Lemma 4.7.3. As a hint, for the $\Longrightarrow$ direction, use Lemma 4.6.1.]

Exercise 29. Let Γ be consistent, and assume that $\Gamma_{all} \cup \{$Some a are $b\} \vdash$ Some p are q. Show that $a \leq_\Gamma p$ or $b \leq_\Gamma p$, and also that $a \leq_\Gamma q$ or $b \leq_\Gamma q$. [Hint: there are two ways to prove this. One is to give a semantic argument: if $a \not\leq_\Gamma p$ and $b \not\leq_\Gamma p$, get a one-point model of $\Gamma_{all} \cup \{$Some a are $b\}$ where $[\![p]\!] = \emptyset$. Alternatively, one can use an induction on proofs in our system.]

5 $\mathcal{R}$: the relational syllogistic

At this point, we turn to logics with verbs. The basic goal is to have a logical system in which we may represent a valid argument such as

$$\frac{\begin{array}{l}\textsf{Every porter recognizes every porter}\\ \textsf{No quarterback recognizes any quarterback}\end{array}}{\textsf{No porter is a quarterback}} \tag{5.1}$$

5.1 Syntax and semantics of $\mathcal{R}$

As in our previous work, we adopt as minimal a syntax as needed. In fact, the English sentences of interest are listed in Figure 5.1. below, using p and q for nouns and r for verbs.

The name of this fragment comes from the word “relation”, since a transitive verb – that is, a verb which takes a direct object – will be interpreted as a relation on the universe M.

It would be possible to write the syntax of $\mathcal{R}$ in English, as we have done with our previous systems. But we prefer to introduce some symbols. The main reason is that some sort of symbolic notation is needed in our larger fragments, if only because the English sentences would not fit on one line in long derivations. And so with an eye to the future, we (somewhat reluctantly) begin now to introduce some notation.

The syntax of $\mathcal{R}$ starts the same way as the syntax of $\mathcal{A}(\mathcal{RC})$ (see page 27). All of our work builds on collections $\mathbf{P}$ of unary atoms and $\mathbf{R}$ of binary atoms. This time we include a complement operation on both unary and binary atoms. (So this is a difference with what we saw earlier with $\mathcal{A}(\mathcal{RC})$.) As in Section 4.4, we understand this operation to be involutive, so $\overline{\overline{p}}$ and p are taken to be identical, as are $\overline{\overline{r}}$ and r. We call the items p, $\overline{p}$, r and $\overline{r}$ *literals.*

check the reference

Set terms At times, it is convenient to adopt a bit of extra notation to the syntax to simplify presentational matters. We introduce *set terms*; these are terms which denote sets in our models. The set terms of $\mathcal{R}$ are

$$p \quad \overline{p} \quad \forall(p,r) \quad \exists(p,r) \quad \forall(p,\overline{r}) \quad \exists(p,\overline{r})$$

and then we can say that the overall syntax of $\mathcal{R}$ is

$$\forall(p,c) \quad \exists(p,c)$$

Set terms play a minor role in this chapter, but when we turn to the larger language $\mathcal{RCA}$ in Section 6, it will be essential to have them.

$\forall(p,q)$	all p are q	$\exists(p,\overline{q})$	some p aren't q
$\exists(p,q)$	some p are q	$\forall(p,\overline{q})$	no p are q
$\forall(p,\forall(q,r))$	all p r all q	$\exists(p,\exists(q,\overline{r}))$	some p don't-r some q
$\forall(p,\exists(q,r))$	all p r some q	$\exists(p,\forall(q,\overline{r}))$	some p don't-r any q
$\exists(p,\forall(q,r))$	some p r all q	$\forall(p,\exists(q,\overline{r}))$	all p don't-r some q
$\exists(p,\exists(q,r))$	some p r some q	$\forall(p,\forall(q,\overline{r}))$	all p don't-r any q

Figure 5.1: The syntax of $\mathcal{R}$, with renderings in English. It's important that our semantics of $\overline{r}$ is the complement relation of the semantics of r, and so this accounts for the unusual (and probably confusing) expression "don't r."

Positive and negative sentences; negations The sentences on the left of Figure 5.1 are called *positive*, and the ones on the right are called *negative*. The way that the figure is arranged, each sentence φ has a *semantic negation* $\overline{\varphi}$ on the other side. For example, if φ is $\forall(p,\overline{q})$, then $\overline{\varphi}$ is $\exists(p,q)$. Note that $\overline{\overline{\varphi}} = \varphi$ for all sentences φ.

Semantics A model $\mathcal{M}$ for $\mathcal{R}$ is a set M, together with an *interpretation* $[\![p]\!] \subseteq M$ for each noun $p \in \mathbf{P}$ and an interpretation $[\![r]\!] \subseteq M^2$ for each verb r. We interpret literals $\overline{p}$ and $\overline{r}$ using complements:

$$[\![\overline{p}]\!] = M \setminus [\![p]\!] \qquad [\![\overline{r}]\!] = M^2 \setminus [\![r]\!].$$

We then interpret set terms by subsets of M in the following way

$$\begin{array}{rcl} [\![\forall(p,s)]\!] & = & \{m \in M : \text{for all } n \in [\![p]\!],\ (m,n) \in [\![s]\!]\} \\ [\![\exists(p,s)]\!] & = & \{m \in M : \text{for some } n \in [\![p]\!],\ (m,n) \in [\![s]\!]\} \end{array}$$

Finally, we have the definition of *truth in a model*:

$$\begin{array}{lll} \mathcal{M} \models \forall(p,c) & \text{iff} & [\![p]\!] \subseteq [\![c]\!] \\ \mathcal{M} \models \exists(p,c) & \text{iff} & [\![p]\!] \cap [\![c]\!] \neq \emptyset \end{array}$$

Finally, we have definitions of semantic consequence relation $\Gamma \models \varphi$, just as we have seen it for other logical languages.

Example 5.1 We consider a simple case, with one unary atom p, and one binary atom s. Consider the following model. We set $M = \{w,x,y,z\}$, and $[\![p]\!] = \{w,x,y\}$. For the relation symbol, s, we take the arrows below:

$$\begin{tikzcd} w \arrow[r] \arrow[dr] & x \arrow[d] \arrow[dl] \\ y \arrow[u] \arrow[r, leftrightarrow] & z \arrow[loop right] \end{tikzcd}$$

For example, $[\![\overline{p}]\!] = \{z\}$, $[\![\forall(p,s)]\!] = \emptyset$, $[\![\exists(\overline{p},s)]\!] = M$, and $[\![\exists(p,\overline{s})]\!] = M$ also. Here are some $\mathcal{R}$-sentences true in $\mathcal{M}$: $\exists(p,p)$ (but note that $\exists(\overline{p},\overline{p})$ is not in the fragment $\mathcal{R}$), and also $\forall(p,\exists(\overline{p},s))$).

Remark Sentences like "All porters recognize some quarterback" are ambiguous in English. In $\mathcal{R}$, we can represent the subject wide scope reading: every porter has the property of recognizing some quarterback or other. It is not possible to represent the object wide scope reading, where there is one particular quarterback who every porter recognizes.

5.2 Algorithmic analysis: building a model of a satisfiable set

In contrast to our other logical systems, we first do the algorithmic analysis, and then turn that analysis into a complete logic for the fragment.

Let Γ be any finite set of sentences of $\mathcal{R}$, satisfiable or not. We aim to build a model $\mathcal{M} = \mathcal{M}(\Gamma)$ with the following properties:

(i) $\mathcal{M}$ satisfies all of the positive sentences in Γ.

(ii) If Γ is satisfiable, then $\mathcal{M} \models \Gamma$.

(iii) $\mathcal{M}$ should be easily computed from Γ.

(iv) If $\mathcal{M} \not\models \Gamma$, then it should be easily traceable to some feature of Γ.

Of course, the second and third statements are vague.

A worked example As you read on, you might also consult Figure 5.2. That is, you might look at the set Γ in Figure 5.2 and see if you can build a model of it in a principled way.

Before we begin the construction, please review the notation $p \xrightarrow[\Gamma]{*} q$ from page **??**.[1] The important semantic point is that if $p \xrightarrow[\Gamma]{*} q$, then $\Gamma \models \forall(p, q)$. (We even have $\Gamma \vdash \forall(p, q)$ in $\mathcal{A}$, but this is irrelevant in this section.)

Since we have only one Γ in this discussion, we lighten the notation and change $\xrightarrow[\Gamma]{*}$ to $\xrightarrow{*}$.

We define the universe of the model together with the interpretations of the atoms simultaneously.

The universe M of the model $\mathcal{M}_\Gamma$ and an associated set M^* of unary atoms.

(i) If $\exists(p, c) \in \Gamma$, then the sentence $\exists(p, c)$ belongs to M, and $p \in M^*$.

(ii) If $\exists(p, q) \in \Gamma$, then in addition $q \in M^*$.

(iii) If $\exists(p, \exists(q, t))) \in \Gamma$, then $q \in M^*$ and q_1, q_2 belong to M.

(iv) If $\forall(p, \exists(q, t)) \in \Gamma$ and $[\![p]\!] \neq \emptyset$, then $q \in M^*$, and q_1, q_2 belong to M.

We use letters like π to denote arbitrary elements of M, keeping in mind that those elements are either existential sentences in M or else unary atoms with subscripts.

[1] I see that there is $\xrightarrow[\Gamma]{*}$ and also $\xrightarrow[\Gamma]{**}$, and I have to straighten this out.

Interpretations $[\![u]\!]$ of unary atoms u in $\mathcal{M}_\Gamma$

(i) If $p \in M^*$ and $p \xrightarrow{*} u$, then $p_1, p_2 \in [\![u]\!]$.

(ii) If $\exists(p, c) \in M$ and $p \xrightarrow{*} u$, then $\exists(p, c) \in [\![u]\!]$.

(iii) If $\exists(p, q) \in M$ and $q \xrightarrow{*} u$, then $\exists(p, q) \in [\![u]\!]$.

Interpretations $[\![r]\!]$ of binary atoms r in $\mathcal{M}_\Gamma$

(i) If $\forall(u, \forall(r, v)) \in \Gamma$, then $x[\![r]\!]y$, for all $x \in [\![u]\!]$ and $y \in [\![v]\!]$.

(ii) If $\forall(u, \exists(q, r)) \in \Gamma$ and $x \in [\![u]\!]$, then ($q_1 \in M$ and) $x[\![r]\!]q_1$.

(iii) If $\exists(p, \exists(q, r)) \in \Gamma$, then $\exists(p, \exists(q, r))[\![r]\!]q_1$.

(iv) If $\exists(p, \forall(q, r)) \in \Gamma$, then $\exists(p, \forall(q, r))[\![r]\!]x$ for all $x \in [\![q]\!]$.

The set M^* We mostly use this set M^* for convenience; it saves a small amount of notation. M^* is the set of unary atoms u such that M contains u_1 and u_2. (Recall that M also contains the existential sentences of Γ.)

Lemma 5.2.1 *If $\varphi \in \Gamma$ and φ is positive, then $\mathcal{M} \models \varphi$.*

Proof We check this by examining all the possible cases as to the syntax of φ. These are all straightforward verifications. For example, let us consider the last part. The sentence $\exists(p, \exists(q, r))$ belongs to M; call it φ. Moreover, $\varphi \in [\![p]\!]$ because $p \xrightarrow{*} p$. In addition, M contains q_1 (and q_2). Finally, $\varphi[\![r]\!]q_1$. This shows that indeed $\mathcal{M} \models \varphi$. ⊣

The converse of the second part of Lemma 5.2.1 is obvious, but the converse of the first part is false. For example, if Γ is $\exists(q, q), \forall(q, p)$, then $M^* = \{q\}$ but $[\![p]\!] = \{q\} \neq \emptyset$.

Our main result is that $\mathcal{M}$ satisfies *all* of the sentences in Γ, *provided it is satisfiable in the first place*. First some subsidiary results:

Lemma 5.2.2 *If $p \in M^*$, then $[\![p]\!] \neq \emptyset$. If $[\![p]\!] \neq \emptyset$, then $\Gamma \models \exists(p, p)$.*

Proof This is by induction on on the construction of the model. ⊣

Lemma 5.2.3 *If $[\![p]\!] \cap [\![q]\!] \neq \emptyset$, then $\Gamma \models \exists(p, q)$.*

Proof There are three cases, depending on the nature of the common element of $[\![p]\!] \cap [\![q]\!]$, and also the reasons for having this in both $[\![p]\!]$ and $[\![q]\!]$. Here are some of the cases.

Suppose that $u_i \in [\![p]\!] \cap [\![q]\!]$. Then $u \in M^*$, and so $\Gamma \models \exists(u, u)$ by Lemma 5.2.2. We also have $u \xrightarrow{*} p, q$ so that $\Gamma \models \forall(u, p)$ and $\Gamma \models \forall(u, q)$. Then easily $\Gamma \models \exists(p, q)$.

Suppose that $\exists(p_1, c_1) \in [\![p]\!] \cap [\![q]\!]$. Then $p_1 \xrightarrow{*} p$ and $p_1 \xrightarrow{*} q$. So $\Gamma \models \forall(p_1, p)$ and $\Gamma \models \forall(p_1, q)$. In this case, $\Gamma \models \exists(p_1, p_1)$, and so we get $\Gamma \models \exists(p, q)$.

For a final case, suppose that $\exists(p_1, q_1) \in [\![p]\!] \cap [\![q]\!]$, say with $p_1 \xrightarrow{*} p$ and $q_1 \xrightarrow{*} q$. Since $\Gamma \models \exists(p_1, p_1)$, and so we once agin get $\Gamma \models \exists(p, q)$. ⊣

Let Γ be the set below

All j are k	All j are l	All p see some k	φ_1: Some n sees some m
All k are l	All l are k	φ_2: Some j is a j	φ_3: Some n fails to see some m
All l are m	All k are n	All l see some n	All l fail to see some n
All m are q	All p are q		
All q are p			

We show $\mathcal{M}_\Gamma$. The set M is

$$\{\varphi_1, \varphi_2, \varphi_3, m_1, m_2, k_1, k_2, n_1, n_2\}$$

The interpretations of the unary atoms are given by

$$\begin{array}{rcl}
[\![p]\!] = [\![q]\!] & = & \{\varphi_2, m_1, m_2, k_1, k_2\} \\
[\![m]\!] & = & \{\varphi_2, m_1, m_2, k_1, k_2\} \\
[\![k]\!] = [\![l]\!] & = & \{\varphi_2, k_1, k_2\} \\
[\![n]\!] & = & \{\varphi_1, \varphi_3, n_1, n_2, k_1, k_2\} \\
[\![j]\!] & = & \{\varphi_2\}
\end{array}$$

The interpretation of see is the set of ordered pairs below:

$$\{(\varphi_1, m_1), (\varphi_2, k_1), (m_1, k_1), (m_2, k_1), (k_1, k_1), (k_2, k_1), (\varphi_2, n_1), (k_1, n_1), (k_2, n_1)\}.$$

Figure 5.2: A worked example. The set Γ includes the set which we saw in Example 2.8 on page 20 .

We know that a set Γ is *unsatisfiable* if there is no model of all of the sentences in Γ. We next introduce a definition of what it means for a set Γ of sentences in $\mathcal{R}$ to be *promptly unsatisfiable*. The definition consists of many cases, and so we list them in Figure 5.3. The idea is that if Γ is promptly unsatisfiable, then it should be "easy" to use one of the conditions to see that Γ really is unsatisfiable; it should not be a matter of "deep reasoning" or extensive searching.

Lemma 5.2.4 *The following are equivalent:*

(i) Γ *is unsatisfiable.*

(ii) Γ *is promptly unsatisfiable.*

(iii) $\mathcal{M} \not\models \Gamma$.

Proof It is trivial that (1)$\Longrightarrow$(3). It is fairly easy to check that (2)$\Longrightarrow$(1): if any of the conditions C1–C6 hold, then Γ is unsatisfiable. Let us check the first of the conditions in Figure 5.3, C1. If $\exists(p, \overline{q})$ gets put in $[\![q]\!]$, it must be that $p \xrightarrow{*} q$. But then $\Gamma \models \forall(p, q)$.

C1 There is a sentence $\exists(p,\overline{q}) \in \Gamma$ such that in $\mathcal{M}$, $\exists(p,\overline{q}) \in [\![q]\!]$.

C2a There is a sentence $\forall(p,\overline{q}) \in \Gamma$ and some $u \in M^*$ so that $u \xrightarrow{*} p$ and $u \xrightarrow{*} q$.

C2b There are $\exists(u,c)$ and $\forall(p,\overline{q})$ in Γ such that in $\mathcal{M}$, $\exists(u,c) \in [\![p]\!] \cap [\![q]\!]$.

C3 The sentence $\forall(p,\forall(q,\overline{r}))$ belongs to Γ, and one of the following hold:

1. There is a sentence $\forall(a,\forall(b,r))$ in Γ such that $[\![p]\!] \cap [\![a]\!] \neq \emptyset$, $[\![q]\!] \cap [\![b]\!] \neq \emptyset$.
2. $\forall(p,\exists(q,r))$ belongs to Γ, and $[\![p]\!] \neq \emptyset$.
3. $\exists(p,\forall(q,r))$ belongs to Γ, and $[\![q]\!] \neq \emptyset$.
4. $\exists(p,\exists(q,r))$ belongs to Γ.

C4a There are $\forall(p,\exists(q,\overline{r}))$ and $\forall(p',\forall(q',r))$ in Γ and $u \in M^*$ such that $u \xrightarrow{*} p, p'$, and $q \xrightarrow{*} q'$.

C4b There are $\forall(p,\exists(q,\overline{r}))$, $\exists(p',\forall(q',r))$ in Γ such that $p' \xrightarrow{*} p$ and $q \xrightarrow{*} q'$.

C5a There are $\exists(p,\forall(q,\overline{r}))$ and $\forall(p',\forall(q',r))$ in Γ, and $x \in [\![q]\!] \cap [\![q']\!]$, such that $p \xrightarrow{*} p'$.

C5b There are $\exists(p,\forall(q,\overline{r}))$ and $\forall(p',\exists(q',r))$ in Γ, such that $p \xrightarrow{*} p'$ and $q' \xrightarrow{*} q$.

C6 There are $\exists(p,\exists(q,\overline{r}))$ and $\forall(u,\forall(v,r))$ in Γ such that $p \xrightarrow{*} u$ and $q \xrightarrow{*} v$.

Figure 5.3: A set Γ is *promptly unsatisfiable* if any of the conditions above hold.

Of course, $\Gamma \models \exists(p,\overline{q})$, since this sentence belongs to Γ. But then $\Gamma \models \exists(q,\overline{q})$. Thus Γ is unsatisfiable.

The main work is to show (3)$\Longrightarrow$(2): assuming that $\mathcal{M} \not\models \Gamma$, we'll show that one of C1–C6 must hold. We do this by examining the six types of *negative* sentences. For each, we assume that it fails in $\mathcal{M}$, and we identify the corresponding condition in Figure 5.3 that must hold.

First, let φ be $\exists(p,\overline{q})$. Note that $\varphi \in M$. As always in this proof, we assume that $\mathcal{M} \not\models \varphi$. In particular, $\varphi \in [\![q]\!]$. Note that $\overline{q}$ is not related by $\xrightarrow{*}$ to anything, so the only way that φ can belong to $[\![q]\!]$ is if $p \xrightarrow{*} q$. Thus C1 holds.

Second, let φ be $\forall(p,\overline{q})$. Since $\mathcal{M} \not\models \varphi$, we have $x \in [\![p]\!] \cap [\![q]\!]$. We have several cases depending on whether x is a sentence φ or a subscripted atom u_i. If x is u_i, then we see that $u \xrightarrow{*} p$ and $u \xrightarrow{*} q$. So we have C2a. If x is $\exists(u,c)$ for some set term c, then C2b holds.

Third, let φ be $\forall(p,\forall(q,\overline{r}))$. Assume that $\mathcal{M} \not\models \varphi$. Let $x \in [\![p]\!]$ be related by $[\![r]\!]$ to $y \in [\![q]\!]$. There are four subcases, corresponding to the four clauses of the definition of $[\![r]\!]$ in $\mathcal{M}$.

Subcase 1: There is a sentence $\forall(a,\forall(b,r))$ in Γ such that $x \in [\![p]\!] \cap [\![a]\!]$, $y \in [\![q]\!] \cap [\![b]\!]$.

Subcase 2: $x \in [\![p]\!]$, y is q_1, and the sentence $\forall(p, \exists(q, r))$ belongs to Γ.
Subcase 3: x is $\exists(p, \forall(q, r))$ and $y \in [\![q]\!]$.
Subcase 4: x is $\exists(p, \exists(q, r))$ and y is q_1.
These together lead to C3.

Fourth, let φ be $\forall(p, \exists(q, \overline{r}))$. Suppose that $\pi \in [\![p]\!]$ is related by $[\![r]\!]$ to all elements of $[\![q]\!]$. Since $\varphi \in \Gamma$, our overall construction has arranged that $q \in M^*$; the point of this is that M contains q_2. We have three subcases concerning π.

Subcase 1: π is of the form u_i, or else $\exists(u, c)$, so that $u \xrightarrow{*} p$, and the reason that $\pi[\![r]\!]q_2$ is that there is a sentence $\forall(p', \forall(q', r)) \in \Gamma$ such that $u \xrightarrow{*} p'$ and $q \xrightarrow{*} q'$. So we have C4a.

Subcase 2: π is of the form $\exists(p', \forall(q', r))$ and also $p' \xrightarrow{*} p$ and $q \xrightarrow{*} q'$. This leads to C4b.

Fifth, let φ be $\exists(p, \forall(q, \overline{r}))$. Assuming that this sentence belongs to Γ, it also belongs to M. Assuming that $\mathcal{M} \not\models \varphi$, we must have some $\pi \in [\![q]\!]$ such that $\varphi[\![r]\!]\pi$.

Case 1: π is x_2 for some $x \in M^*$. In this case, Γ contains a sentence $\forall(p', \forall(q', r))$ such that $x \xrightarrow{*} q, q'$ and $p \xrightarrow{*} p'$. This leads to C5a.

Case 2: π is x_1 for some $x \in M^*$. One possibility is (as in Case 1 just above) Γ contains a sentence $\forall(p', \forall(q', r))$ such that $x \xrightarrow{*} q, q'$ and $p \xrightarrow{*} p'$. This again leads to C5a. The other possibility is that $\forall(p', \exists(q, r))$, and $p \xrightarrow{*} p'$. This leads to C5b.

Case 3: π is a sentence $\exists(q', c)$. In this case, there is a sentence $\forall(p', \forall(q'', r))$ such that $p \xrightarrow{*} p'$, $q' \xrightarrow{*} q''$ and $q' \xrightarrow{*} q$. This also implies C5a.

Finally, let φ be $\exists(p, \exists(q, \overline{r}))$. Assume that $\mathcal{M} \not\models \varphi$. Then $\varphi \in M$ and also $q_2 \in M$. We have $\varphi \in [\![p]\!]$ and $q_2 \in [\![q]\!]$. Then $\varphi[\![r]\!]q_2$. And so there is some sentence $\forall(u, \forall(v, r)) \in \Gamma$ such that $\varphi \in [\![u]\!]$ and $q_2 \in [\![v]\!]$. This means that $p \xrightarrow{*} u$ and $q \xrightarrow{*} v$. So C6 holds.

This completes the proof. ⊣

Lemma 5.2.5 *holds. The following are equivalent:* Check this!

(i) Γ *is satisfiable.*

(ii) $\mathcal{M}_\Gamma \models \Gamma$.

Lemma 5.2.5 is just a restatement of Lemma 5.2.4.

Summary: an algorithm for the consequence relation of $\mathcal{R}$ Up until now, we have taken a set Γ and built a model of it, assuming that Γ has a model in the first place. One point of the work we have done concerns complexity: the procedure to build $\mathcal{M}_\Gamma$ from Γ is efficient; its time complexity is polynomial in the size of Γ. A second point is that another equally interesting question is *reducible* to the question of whether a set is satisfiable or not. We have in mind the *consequence question*: given (a finite set) Γ and φ, tell whether or not $\Gamma \models \varphi$. The reason that this is reducible to the question that we have already answered is that

$$\Gamma \models \varphi \quad \text{iff} \quad \Gamma \cup \{\overline{\varphi}\} \text{ has no model.}$$

(i) Let $\Delta = \Gamma \cup \{\overline{\varphi}\}$.

(ii) Build the model $\mathcal{M} = \mathcal{M}_\Delta$ as in this section.

(iii) Check to see whether $\mathcal{M}$ satisfies all the sentences in Γ.
   a) If 'no', then Γ is unsatisfiable, and so $\Gamma \models \varphi$ vacuously.
   b) If 'yes', then see if in addition $\mathcal{M} \models \overline{\varphi}$. If 'yes', then we have a model of Δ, and so $\Gamma \not\models \varphi$. If 'no', then $\Gamma \models \varphi$.

Figure 5.4: An algorithm to see if $\Gamma \models \varphi$ in $\mathcal{R}$.

(You should be sure you understand this quite well.) Figure 5.4 summarizes this discussion in the form of an algorithm for the consequence question.

5.3 Proof theory

We have already seen an analysis of the relation $\Gamma \models \varphi$, and in particular we have seen several important results concerning this relation. At this time, we wish to turn our previous results into a sound and complete logical system for $\mathcal{R}$. The leading idea at this point is to have whatever rules it takes to "convert $\models$ to $\vdash$". That is, we aim to have enough rules to insure facts like the following:

(i) If $p \overset{*}{\to} q$, then $\Gamma \vdash \forall(p, q)$.

(ii) If Γ is promptly unsatisfiable, then $\Gamma \vdash \bot$.

(iii) $\Gamma \vdash \varphi$ iff $\Gamma \cup \{\overline{\varphi}\} \vdash \bot$.

The first requirement is met by incorporating the logic for $\mathcal{A}$ into our system. The second is met by going carefully through the proofs above, and adding whatever rules it takes. Of course, it is better to adopt a set of rules which "looks nice", and so this is what we have tried to do. Finally, the last requirement is met by adopting a rule of *reductio ad absurdum*, as we discuss below.

Our rules appear in Figure 5.5. We remind the reader that p and q range over unary atoms, c over set terms, and t over binary literals.

Rules (D1), (D2), (D3), (B), (A), (T) and (I) are natural generalizations of their namesakes in S. In contrast, ($\forall\forall$), ($\exists\exists$), ($\forall\exists$) and (II) express genuinely relational logical principles. In some settings, these last rules are called *monotonicity principles*.

In addition to syllogistic rules, R contains the rule of (RAA), whereby one derives φ from Γ by temporarily adding the "negation" of φ to Γ, thereby obtaining $\Gamma \cup \{\overline{\varphi}\}$, and then deriving a contradiction from this larger set $\Gamma \cup \{\overline{\varphi}\}$. The idea is that This rule (RAA) is *not the same* as *ex falso quodlibet*, the rule we have called (X) in Section 4.4. To see why, we mention the semantic justifications for both rules.

$$\frac{\exists(p,q) \qquad \forall(q,c)}{\exists(p,c)}\ (\mathrm{D1}) \qquad \frac{\forall(p,q) \qquad \forall(q,c)}{\forall(p,c)}\ (\mathrm{B})$$

$$\frac{\forall(p,q) \qquad \exists(p,c)}{\exists(q,c)}\ (\mathrm{D2}) \qquad \frac{}{\forall(p,p)}\ (\mathrm{T}) \qquad \frac{\exists(p,c)}{\exists(p,p)}\ (\mathrm{I})$$

$$\frac{\forall(q,\bar{c}) \qquad \exists(p,c)}{\exists(p,\bar{q})}\ (\mathrm{D3}) \qquad \frac{\forall(p,\bar{p})}{\forall(p,c)}\ (\mathrm{A}) \qquad \frac{\exists(p,\exists(q,t))}{\exists(q,q)}\ (\mathrm{II})$$

$$\frac{\forall(p,\forall(q',t)) \qquad \exists(q,q')}{\forall(p,\exists(q,t))}\ (\forall\forall) \qquad \frac{\exists(p,\exists(q,t)) \qquad \forall(q,q')}{\exists(p,\exists(q',t))}\ (\exists\exists)$$

$$\frac{\forall(p,\exists(q,t)) \qquad \forall(q,q')}{\forall(p,\exists(q',t))}\ (\forall\exists) \qquad \frac{\begin{array}{c}[\overline{\varphi}]\\ \vdots \\ \bot\end{array}}{\varphi}\ \mathrm{RAA}$$

Figure 5.5: The logical system R for $\mathcal{R}$. Notice the rule of (RAA).

The bottom symbol $\bot$ We also define $\bot$ to be any contradiction. In **R**, this means a sentence of the form $\exists(p, \overline{p})$.

Here are the ideas behind (RAA) and (X):

(i) If $\Gamma \cup \{\overline{\varphi}\} \models \bot$, then $\Gamma \models \varphi$.

(ii) If $\Gamma \models \bot$, then $\Gamma \models \varphi$

The (RAA) justification changes the assumptions in the derivation from $\Gamma \cup \{\overline{\varphi}\}$ to Γ. This is the exact difference between *reductio ad absurdum* and *ex falso quodlibet.* 73

We must incorporate this observation into our proof system, and we do so in (RAA). We now can display this rule in natural-deduction-style:

$$\frac{\begin{array}{c}[\overline{\varphi}]\\ \vdots \\ \bot\end{array}}{\varphi}\ \mathrm{RAA}$$

Definition A *proof tree over* Γ is a pair $(\mathcal{T}, \mathrm{Can})$, where finite tree $\mathcal{T}$ whose nodes are labeled with sentences, and Can is a set of labeled leaves of $\mathcal{T}$ called the *canceled leaves.* Each node n in the tree must satisfy one of the following conditions:

(i) n is a leaf labeled by an element of Γ.

(ii) n comes from its parent(s) by an application of a rule other than (RAA).

(iii) $n \in \mathrm{Can}$, and there is some node m on the path from n to the root such that the parent of m is labeled $\perp$, and the label of m is the semantic negation of the label of n.

We write $\Gamma \vdash \varphi$ if there is a proof tree $\mathcal{T}$ with φ at the root whose uncanceled leaves all belong to Γ.

Example 5.2 Here is a derivation showing that

$$\forall(x, \overline{x}) \vdash \forall(y, \forall(x, r))$$

In words, if there are no xs, then all y's have any relation whatsoever to all of them. Note that we cannot simply use the rule

$$\frac{\forall(p, \bar{p})}{\forall(p, c)}\ (\mathrm{A})$$

Instead, we use RAA:

$$\dfrac{\dfrac{\dfrac{[\exists(y, \exists(x, \overline{r}))]^1}{\exists(x, x)} \qquad \forall(x, \overline{x})}{\exists(x, \overline{x})}}{\forall(y, \forall(x, r))}\ (\mathrm{RAA})^1$$

This example also shows that we indicate canceled leaves using bracketing, and we also use numerical superscripts to tell which application of (RAA) has canceled which leaves.

Example 5.3 Here is a derivation showing that

$$\forall(x, \forall(y, r)), \forall(p, y), \exists(p, q) \vdash \forall(x, \exists(y, r))$$

$$\dfrac{\dfrac{\forall(x, \forall(y, r)) \qquad \dfrac{\exists(p, q) \quad \forall(p, y)}{\exists(p, y)}\ (\mathrm{D1})}{\forall(x, \exists(p, r))}\ (\forall\forall) \qquad \forall(p, y)}{\forall(x, \exists(y, r))}\ (\forall\exists)$$

Example 5.4 Here is a formal proof showing that

$$\forall(x,\overline{x}) \vdash \forall(y,\forall(x,r))$$

In words, if there are no xs, then all y's have any relation whatsoever to all of them. As in Example 5.2, this does not follow from the rule (A). But here is a derivation:

$$\dfrac{\dfrac{\dfrac{[\exists(y,\exists(x,\overline{r}))]^1}{\exists(x,x)}\,(\mathrm{II}) \qquad \forall(x,\overline{x})}{\exists(x,\overline{x})}\,(\mathrm{D1})}{\forall(y,\forall(x,r))}\,(\mathrm{RAA})^1$$

As we shall see below in Theorem 9.2.1, the rule (RAA) is essential.

Although (RAA) may be used at any point in a derivation, our proof system in this section has the extra property that if $\Gamma \vdash \varphi$ using (RAA), then there is a proof using (RAA) at most once. In this book we are not going to be concerned with this stronger check!
property, and for more on it see [**?**]. check this

Now we have stated the rules of our system, and the natural question is to whether they are complete. We assert without proof that they are. To check this completeness result, one would have to go back to our work in Section 4.7 and to be sure that all of the work holds when we change the semantic assertions such as "$\Gamma \models \varphi$" and "Γ is satisfiable" to their syntactic variants "$\Gamma \vdash \varphi$" and "Γ is consistent." We are not going to do this in detail, but for the record we state the following result.

Theorem 5.3.1 (Completeness of the proof system for $\mathcal{R}$) $\Gamma \vdash \varphi$ *iff* $\Gamma \models \varphi$.

5.4 Incorporating background facts in $\mathcal{R}$

Suppose we have a stock of background facts about verbs, such as

$$\text{hitting entails touching} \tag{5.2}$$

We mean that every act of hitting is also an act of touching. This background fact cannot be stated in any of the languages which we have so far studied. Nevertheless, it can be made into a semantic requirement: we would require of a model that the interpretation of *hitting* be a sub-relation of the interpretation of *touching*. And then we might like to study the entailment relation on this smaller class of models.

Even though we cannot state (5.2) as an *axiom*, it does yield a rule of inference. To state it more abstractly, suppose we have a rule like

$$r \Rightarrow s \tag{5.3}$$

and we restrict attention to the models where $[\![r]\!] \subseteq [\![s]\!]$. Then Figure 5.6 lists sound rules of inference.

We add these to the system $\mathcal{R}$ as listed in Figure 5.5.

$$\frac{\forall(d,\forall(c,r))}{\forall(d,\forall(c,s))} \qquad \frac{\forall(d,\exists(c,r))}{\forall(d,\exists(c,s))} \qquad \frac{\exists(d,\forall(c,r))}{\exists(d,\forall(c,s))} \qquad \frac{\exists(d,\exists(c,r))}{\exists(d,\exists(c,s))}$$

Figure 5.6: Rules of inference corresponding to the background assertion $r \Rightarrow s$.

Proposition 5.4.1 *The system* R *together with the rules in Figure 5.6 give a sound and complete logic:* $\Gamma \vdash \varphi$ *iff* $\Gamma \models \varphi$.

this needs to be added, either as a result in the text or as an exercise

Sources The first syllogistic logic to employ verbs was Nishihara, Morita, and Iwata [**?**]. The logical system for $\mathcal{R}$ comes from Moss [**?**] and from Pratt-Hartmann and Moss [**?**]. The work on incorporating background facts is new here.

5.5 Exercises

Exercise 30. Let Γ be the set of sentences in Figure 5.2 and shown again below:

All j are k	All j are l	All k are l	All p see some k	φ_1: Some n sees some m
All l are k	All l are m	All k are n	φ_2: Some j is a j	φ_3: Some n fails to see some m
All m are q	All p are q	All q are p	All l see some n	All l fail to see some n

For each of the following sentence φ, do the following: if $\Gamma \vdash \varphi$, then give a derivation in our system. if $\Gamma \not\vdash \varphi$, then give a model of Γ where φ is false.

(i) Some p sees some q

(ii) All p see all q

(iii) Some k sees some p

(iv) All k fail to see some p

Exercise 31. In this section we gave a precise definition of the derivability relation $\Gamma \vdash \varphi$. Here is another characterization of this relation. Let $\vdash_{alt}$ be the the smallest relation on $\mathcal{P}(\mathcal{R}) \times \mathcal{R}$ such that

(i) If $\varphi \in \Gamma$, then $\Gamma \vdash_{alt} \varphi$.

(ii) If $\Gamma \vdash_{alt} \psi_1, \ldots, \Gamma \vdash_{alt} \psi_n$ and one of the rules of R other than (RAA) has as a substitution instance $\psi_1, \ldots, \psi_n \backslash \varphi$, then $\Gamma \vdash_{alt} \varphi$.

(iii) If $\Gamma \cup \{\overline{\varphi}\} \vdash_{alt} \bot$, then $\Gamma \vdash_{alt} \varphi$.

Prove that $\Gamma \vdash \varphi$ iff $\Gamma \vdash_{alt} \varphi$.

6 $\mathcal{RCA}$: verbs, relative clauses, and comparative adjectives

We started out with the smallest logical system in the world, the syllogistic system of All. At this point, we wish to consider one of the largest syllogistic systems of all, the system $\mathcal{RCA}$. This system has transitive verbs, subject relative clauses, and capable of representing a fairly large class of natural language inferences.

Here is an of the kind of inference we want to do in this chapter.

Example 6.1

$$\frac{\begin{array}{l}\textsf{Every hyena is taller than some jackal}\\ \textsf{Everything taller than some jackal is not heavier than any warthog}\end{array}}{\textsf{Everything which is taller than some hyena is not heavier than any warthog}}$$

6.1 Syntax and semantics

The syntax of the language $\mathcal{RCA}$ is shown in Figure 6.1. We start with one collection of *unary atoms* (for nouns), another collection of *tv atoms* (for transitive verbs), and finally a third collection of *adjective atoms* (for comparative adjectives). The second column in the figure indicates the variables that we shall use in order to refer to the objects of the various syntactic categories. Because the syntax is not standard, it will be worthwhile to go through it slowly and to provide glosses in English for expressions of various types.

Our intention is that unary atoms represent plural nouns, adjective atoms represent comparative adjective phrases such as larger than and smaller than, and tv atoms represent transitive verbs. We group the adjective atoms and tv atoms into binary atoms, and r we use letters like r for those. Moving on, we have *set terms*; these are named because in the semantics they denote sets. To understand how they work, let us exhibit a rendering of the simplest set terms into more idiomatic English:

$\forall(\mathsf{boy}, \mathsf{see})$	those who see all boys
$\exists(\mathsf{girl}, \mathsf{taller})$	those who are taller than some girl(s)
$\forall(\mathsf{boy}, \overline{\mathsf{see}})$	those who fail-to-see all boys
	= those who see no boys
	= those who don't see any boys
$\exists(\mathsf{girl}, \overline{\mathsf{see}})$	those who fail-to-see some girl
	= those who don't see some girl

expression	variables	syntax
unary atom	p, q	
adjective atom	a	
tv atom	v	
binary atom	r	$v \mid a$
positive set term	c^+, d^+	$p \mid \exists(p,r) \mid \forall(p,r)$
set term	c, d	$c^+ \mid \overline{p} \mid \exists(p,\overline{r}) \mid \forall(p,\overline{r})$
sentence	φ	$\forall(d^+,c) \mid \exists(d^+,c)$

Figure 6.1: The syntax of $\mathcal{RCA}$.

The bar notation indicates negation. The semantics will work "classically" in the sense that the interpretations of cat and $\overline{\text{cat}}$ will be set complements; this is a choice that could be reconsidered, of course. Returning to set terms and how we read them, the syntax indicates that the set terms in this language are a recursive construct. That is, we may embed set terms. So we have set terms like

$$\exists(\forall(\mathsf{cat}, \overline{\mathsf{sees}}), \mathsf{taller})$$

which may be taken to denote the individuals who are taller than someone who sees no cat.

We should note that the relative clauses which can be obtained in this way are all "missing the subject", never "missing the object". The language is too poor to express predicates like $\lambda x.$all boys see x.

We also have sentences using the constants, such as $\forall(g, s)(m)$, corresponding to Mary sees all girls. But we are not able to say all girls see Mary; the syntax again is too weak. We should note that the relative clauses which can be obtained in this way are all "missing the subject", never "missing the object". The language is too poor to express predicates like $\lambda x.$all boys see x.

The bar notation. We intend the bar notation for negation to be involutive: that is, we shall never iterate the bar notation, and so we simply will not see expressions such as $\overline{\overline{p}}$. That is, we make the choice that our syntax does not include such a notation. However, it will be convenient to adopt a syntactic negation and to say that the syntax should be closed under the bar notation; this means that we would be *identifying* $\overline{\overline{p}}$ with p, and making similar identifications.

A *positive* set term is either a unary atom, or else a quantified set term with a non-negated verb or adjective. More generally, a *set term* also allows negation, indicated by the overline on the atom.

The sentences in the language are of the form $\forall(b, c)$ and $\exists(b, c)$; they can be read as statements of the inclusion of one set term extension in another, and of the non-empty intersection. The first must be a *positive* set term.

The bar notation in general. We have already seen that our unary and binary atoms come with negative forms. We extend this notation to set terms and sentences in the following ways: $\overline{\overline{p}} = p$, $\overline{\overline{s}} = s$, $\overline{\exists(l,r)} = \forall(l,\overline{r})$, $\overline{\forall(l,r)} = \exists(l,\overline{r})$, $\overline{\forall(c,d)} = \exists(c,\overline{d})$, and $\overline{\exists(c,d)} = \forall(c,\overline{d})$.

We call φ and $\bar{\varphi}$ *semantic negations* for the following reason: for all models $\mathcal{M}$, $\mathcal{M} \models \varphi$ iff $\mathcal{M} \not\models \bar{\varphi}$.

Semantics. A *model* (for this language $\mathcal{L}$) is a pair $\mathcal{M} = (M, [\![\ ]\!])$, where M is a non-empty set, $[\![p]\!] \subseteq M$ for all $p \in \mathbf{P}$, $[\![r]\!] \subseteq M^2$ for all binary atoms $r \in \mathbf{R}$. The only requirement is that for all adjectives a, $[\![a]\!]$ should be a *transitive* relation: if $a(x,y)$ and $a(y,z)$, then $a(x,z)$.

Given a model $\mathcal{M}$, we extend the interpretation function $[\![\ ]\!]$ to the rest of the language by setting

$$\begin{array}{lcl} [\![\overline{p}]\!] & = & M \setminus [\![p]\!] \\ [\![\overline{r}]\!] & = & M^2 \setminus [\![r]\!] \\ [\![\exists(l,t)]\!] & = & \{x \in M : \text{ for some } y \text{ such that } [\![l]\!](y),\ [\![t]\!](x,y)\} \\ [\![\forall(l,t)]\!] & = & \{x \in M : \text{ for all } y \text{ such that } [\![l]\!](y),\ [\![t]\!](x,y)\} \end{array}$$

We define the truth relation $\models$ between models and sentences by:

$$\begin{array}{lcl} \mathcal{M} \models \forall(c,d) & \text{iff} & [\![c]\!] \subseteq [\![d]\!] \\ \mathcal{M} \models \exists(c,d) & \text{iff} & [\![c]\!] \cap [\![d]\!] \neq \emptyset \end{array} \tag{6.1}$$

If Γ is a set of formulas, we write $\mathcal{M} \models \Gamma$ if for all $\varphi \in \Gamma$, $\mathcal{M} \models \varphi$.

Example 6.2 Consider the model $\mathcal{M}$ with $M = \{w,x,y,z\}$, $[\![\mathsf{cat}]\!] = \{w,x,y\}$, $[\![\mathsf{dog}]\!] = \{z\}$, with $[\![\mathsf{see}]\!]$ shown below on the left, and $[\![\mathsf{bigger}]\!]$ on the right:

$$\begin{tikzcd} w \arrow[dr] & x \arrow[l] \arrow[d] \arrow[dl] \\ y \arrow[u] \arrow[r, leftrightarrow] & z \arrow[loop right] \end{tikzcd} \qquad\qquad x, z \longrightarrow y \longrightarrow w$$

Then $[\![\exists(\mathsf{dog},\mathsf{see})]\!]$ is the set of entities that see some dog, namely $\{x,y,z,w\}$. Similarly, $[\![\exists(\mathsf{dog},\mathsf{bigger})]\!] = \{w,y\}$. It follows that

$$[\![\forall(\exists(\mathsf{dog},\mathsf{bigger}),\mathsf{see})]\!] = \{x\}.$$

Since $[\![\mathsf{cat}]\!]$ contains x, we have

$$\mathcal{M} \models \forall(\forall(\exists(\mathsf{dog},\mathsf{bigger}),\mathsf{see}),\mathsf{cat}).$$

That is, in our model it is true that everything which sees everything bigger than some dog is a cat.

$$\frac{}{\forall(c,c)}\ (\mathrm{T}) \qquad \frac{\exists(c,d)}{\exists(c,c)}\ (\mathrm{I}) \qquad \frac{\forall(c,\bar{b})}{\forall(b,\bar{c})}\ (\mathrm{C}) \qquad \frac{\forall(b,c) \qquad \forall(c,d)}{\forall(b,d)}\ (\mathrm{B})$$

$$\frac{\exists(b,c) \qquad \forall(c,d)}{\exists(b,d)}\ (\mathrm{D1}) \qquad \frac{\forall(b,c) \qquad \exists(b,d)}{\exists(c,d)}\ (\mathrm{D2})$$

$$\frac{\forall(p,q)}{\forall(\forall(q,r),\forall(p,r))}\ (\mathrm{J}) \qquad \frac{\forall(p,q)}{\forall(\exists(p,r),\exists(q,r))}\ (\mathrm{K})$$

$$\frac{\exists(p,q)}{\forall(\forall(p,r),\exists(q,r))}\ (\mathrm{L}) \qquad \frac{\exists(q,\exists(p,r))}{\exists(p,p)}\ (\mathrm{II})$$

$$\frac{\forall(p,\bar{p})}{\forall(c,\forall(p,r))}\ (\mathrm{Z}) \qquad \frac{\forall(p,\bar{p})}{\exists(\forall(p,r),\forall(p,r))}\ (\mathrm{W})$$

$$\frac{\forall(p,\exists(q,a))}{\forall(\exists(p,a),\exists(q,a))}\ (\mathrm{tr1}) \qquad \frac{\forall(p,\forall(q,a))}{\forall(\exists(p,a),\forall(q,a))}\ (\mathrm{tr2})$$

$$\frac{\exists(p,\forall(q,a))}{\forall(\forall(p,a),\forall(q,a))}\ (\mathrm{tr3}) \qquad \frac{\exists(p,\exists(q,a))}{\forall(\forall(p,a),\exists(q,a))}\ (\mathrm{tr4})$$

$$\begin{array}{c}[\overline{\varphi}] \\ \vdots \\ \dfrac{\bot}{\varphi}\ \mathrm{RAA}\end{array}$$

Figure 6.2: The proof system for $\mathcal{RCA}$. In it, p and q range over unary atoms, b and c over set terms, d over positive set terms, r over binary atoms, and a over adjective atoms.

Example 6.3 The following putative inference is invalid:

$$\begin{array}{l}
\textsf{Every giraffe sees every gnu} \\
\textsf{Some gnu sees every lion} \\
\textsf{Some lion sees some zebra} \\
\hline
\textsf{Every giraffe sees some zebra}
\end{array}$$

To see this, consider the model shown below

$$\text{giraffe} \longrightarrow \text{gnu} \longrightarrow \text{lion} \longrightarrow \text{zebra}$$

The interpretations of the unary atoms are obvious, and the interpretation of the verb is the relation indicated by the arrow.

6.2 Proof system

At this point, we have a syntax and a semantics. Then we have a notion of *semantic consequence* $\Gamma \models \varphi$, where Γ is a set of sentences in the current fragment, and φ is also a sentence in it. As always, this means that every model of all sentences in Γ is also a model of φ. We give the rules of a natural deduction proof system for validity in this fragment in Figure 6.2. The system generates trees, just as in the proof system of Section 2 but with many more rules. The majority of the rules of the system (those above the last four) are the system for the logic $\mathcal{R}^*$ in [?]. The first two lines are syllogistic rules. We can read an instance of (C): if no senators are millionaires, then no millionaires are senators. We need both (D) rules because of the restriction that d must be a positive set term. Here are some readings the other rules:

(J) If all watches are gold items, then everyone who owns all gold items owns all watches.

(K) If all watches are gold items, then everyone who owns some watch owns some gold item.

(L) If some watches are gold items, then everyone who owns all watches owns some gold item.

(II) If someone owns a watch, then there is a watch.

(tr1) If all watches are bigger than some pencil, then everthing bigger than some watch is bigger than some pencil.

(tr2) If all watches are bigger than all pencils, then everthing bigger than some watch is bigger than all pencils.

(tr3) If some watch is bigger than all pencils, then everthing bigger than all watches is bigger than all pencils.

(tr4) If some watch is bigger than some pencil, then everthing bigger than all watches is bigger than some pencil.

Notice that the validity of the four rules is due to the transitivity of comparative adjectives. These rules come from [**?**]. The overall import of this logical system is that it is complete: every valid inference in the language of this fragment may be syntactically derived in the proof system.

check!

Example 6.4 Here is a derivation for (1.3) from early on in this book:

$$\dfrac{\dfrac{\forall(\mathsf{skunk},\mathsf{mammal})}{\forall(\forall(\mathsf{mammal},\mathsf{respect}),\forall(\mathsf{skunk},\mathsf{respect}))}\ (\mathrm{J})}{\forall(\forall(\forall(\mathsf{skunk},\mathsf{respect}),\mathsf{fear}),\forall(\forall(\mathsf{mammal},\mathsf{respect}),\mathsf{fear}))}\ (\mathrm{J})$$

Note that inference using (J) is *antitone* each time: skunk and mammal have switched positions.

Example 6.5 Here is an example of a derivation which formalizes the reasoning in Example 6.1:

$$\dfrac{\dfrac{\forall(\mathsf{hyena},\exists(\mathsf{jackal},\mathsf{taller}))}{\forall(\exists(\mathsf{hyena},\mathsf{taller}),\exists(\mathsf{jackal},\mathsf{taller}))}\ (\mathrm{tr1}) \quad \forall(\exists(\mathsf{jackal},\mathsf{taller}),\forall(\mathsf{warthog},\overline{\mathsf{heavier}}))}{\forall(\exists(\mathsf{hyena},\mathsf{taller}),\forall(\mathsf{warthog},\overline{\mathsf{heavier}})))}\ (\mathrm{B})$$

The application of (tr1) corresponds to using the premise $\mathsf{every\ hyena\ is\ taller\ than\ some\ jackal}$ to derive the sentence $\mathsf{everything\ which\ is\ taller\ than\ some\ hyena\ is\ taller\ than\ some\ jackal}$. The second step corresponds to the transitivity of predication ($\mathsf{is\ taller\ than}$).

Reductio ad absurdum In addition to syllogistic rules, the logic contains the rule of *reductio ad absurdum* (RAA), whereby one derives φ from Γ by temporarily adding the "negation" of φ to Γ, thereby obtaining $\Gamma\cup\{\overline{\varphi}\}$, and then deriving a contradiction from this larger set $\Gamma\cup\{\overline{\varphi}\}$.

This rule (RAA) is *not the same* as *ex falso quodlibet*, the rule we have called (X) in Section 4.2 (see page 41). To see why, we mention the semantic justifications for both rules.

The bottom symbol $\bot$ We define $\bot$ to be any contradiction. In **R**, this means a sentence of the form $\exists(p, \overline{p})$.

Here are the ideas behind (RAA) and (X):

(i) If $\Gamma \cup \{\overline{\varphi}\} \models \bot$, then $\Gamma \models \varphi$.

(ii) If $\Gamma \models \bot$, then $\Gamma \models \varphi$

The (RAA) justification changes the assumptions in the derivation from $\Gamma \cup \{\overline{\varphi}\}$ to Γ. This is the exact difference between *reductio ad absurdum* and *ex falso quodlibet*.

We must incorporate this observation into our proof system, and we do so in (RAA). We now can display this rule in natural-deduction-style:

$$\dfrac{\begin{array}{c}[\overline{\varphi}]\\ \vdots\\ \bot\end{array}}{\varphi}\ \text{RAA}$$

Definition A *proof tree over* Γ is a pair $(\mathcal{T}, \text{Can})$, where $\mathcal{T}$ is a finite tree whose nodes are labeled with sentences, and Can is a set of labeled leaves of $\mathcal{T}$ called the *canceled leaves*. Each node n in the tree must satisfy one of the following conditions:

(i) n is a leaf labeled by an element of Γ.

(ii) n comes from its parent(s) by an application of a rule other than (RAA).

(iii) $n \in \text{Can}$, and there is some node m on the path from n to the root such that the parent of m is labeled $\bot$, and the label of m is the semantic negation of the label of n.

We write $\Gamma \vdash \varphi$ if there is a proof tree $\mathcal{T}$ with φ at the root whose uncanceled leaves all belong to Γ.

Example 6.6 Here is a proof of the (Zero) rule of $\mathcal{S}^\dagger$: $\forall(p, \overline{p}) \vdash \forall(p, q)$:

$$\dfrac{\dfrac{\dfrac{[\exists(p,\overline{q})]^1}{\exists(p,p)}\ (\text{I}) \qquad \forall(p,\overline{p})}{\exists(p,\overline{p})}\ (\text{D1})}{\forall(p,q)}\ (\text{RAA})^1$$

Example 6.7 Similarly, here is a formal proof showing (A) from the system for $\mathcal{R}$ in Figure 5.5 on page 63:

$$\forall(p, \overline{p}) \vdash \forall(y, \forall(p, d))$$

Here is a derivation:

$$\dfrac{\dfrac{\dfrac{[\exists(p, \overline{d})]^1}{\exists(p, p)}\,(\mathrm{II}) \qquad \forall(p, \overline{p})}{\exists(p, \overline{p})}\,(\mathrm{D1})}{\forall(p, d)}\,(\mathrm{RAA})^1$$

This example also shows that we indicate canceled leaves using bracketing, and we also use numerical superscripts to tell which application of (RAA) has canceled which leaves.

As we shall see below, the special status of (RAA) is essential: indirect syllogistic systems are in general more powerful than direct syllogistic systems.

Although (RAA) may be used at any point in a derivation, our proof system in this section has the extra property that if $\Gamma \vdash \varphi$ using (RAA), then there is a proof using (RAA) at most once. In this book we are not going to be concerned with this stronger property, and for more on it see [**?**].

Check this

6.3 Completeness of the logic

At this point, we mention a few features of the logical system which play a role in our completeness proof. The first is *proof by cases.*

Lemma 6.3.1 (Cases) *Suppose that* $\Gamma \cup \{\varphi\} \vdash \psi$ *and also* $\Gamma \cup \{\overline{\varphi}\} \vdash \psi$*. Then* $\Gamma \vdash \psi$*.*

Proof The hypotheses imply that $\Gamma \cup \{\varphi, \overline{\psi}\} \vdash \bot$. Also $\Gamma \cup \{\overline{\varphi}, \overline{\psi}\} \vdash \bot$, and by (RAA) we also have $\Gamma \cup \{\overline{\psi}\} \vdash \varphi$. Let $\mathcal{T}_1$ be a proof tree showing that $\Gamma \cup \{\varphi, \overline{\psi}\} \vdash \bot$, let $\mathcal{T}_2$ be a proof tree showing that $\Gamma \cup \{\overline{\psi}\} \vdash \varphi$. Take $\mathcal{T}_1$ and replace every leaf labeled φ with a copy of $\mathcal{T}_2$. This gives a proof tree with root $\bot$ and whose leaves are labeled in $\Gamma \cup \{\overline{\psi}\} \vdash \bot$. Using (RAA) again, we see that $\Gamma \vdash \psi$. $\dashv$

Definition A set Γ is *complete* if for all φ, either $\varphi \in \Gamma$ or $\overline{\varphi} \in \Gamma$.[1]

Lemma 6.3.2 *Every consistent set* Γ *has a complete and consistent superset* Γ^**.*

Proof We are going to use Zorn's Lemma[2]. Let $\mathcal{C}$ be the set of consistent $\Delta \supseteq \Gamma$, ordered by inclusion. A *chain* in $\mathcal{C}$ is a set X of elements of $\mathcal{C}$ with the property that if

[1] Please do not confuse the completeness of a set of sentences with the completeness of the logical system under discussion.

[2] If you are not comfortable with Zorn's Lemma, it is also possible to prove the result in the case that the overall language is countable by a step-by-step procedure. Here is a sketch. One lists the sentences in the language in a list as φ_0, φ_1, ..., φ_n, ... One also constructs an infinite sequence $\Gamma = \Gamma_0 \subseteq \cdots \Gamma_n \subseteq \cdots$ of sets of sentences. Γ_{n+1} is either $\Gamma_n \cup \{\varphi_n\}$ or else $\Gamma_n \cup \{\overline{\varphi}_n\}$, whichever is consistent. One must be consistent, lest Γ_n be inconsistent by Lemma 6.3.1. And then $\bigcup_n \Gamma_n$ would be as desired.

X contains both Δ_1 and Δ_2, then either $\Delta_1 \subseteq \Delta_2$ or $\Delta_2 \subseteq \Delta_1$. To use Zorn's Lemma, we must check that every chain X in $\mathcal{C}$ has an upper bound. This is immediate: we take the union $\bigcup X$. The important point is that this is a consistent set, and this point follows from the fact that proofs are finite.

Zorn's Lemma applies and gives us a maximal element Γ^* of $\mathcal{C}$. Γ^* is thus a consistent superset of Γ. We need only check that it is complete. For if not, suppose that neither φ nor $\overline{\varphi}$ belong to Γ^*. By maximality, both $\Gamma^* \cup \{\varphi\} \vdash \bot$ and $\Gamma^* \cup \{\overline{\varphi}\} \vdash \bot$. By Lemma 6.3.1, $\Gamma^* \vdash \bot$. This contradicts the consistency of Γ^*, and so we conclude that Γ^* is indeed complete. $\dashv$

A logic has a *semantic negation*, let $\overline{\varphi}$ be such that every model satisfies exactly one of them.

Lemma 6.3.3 (A useful sufficient condition for completeness) *Let $\mathcal{L}$ be a logic with a semantic negation, and let $\vdash$ be a proof system with* RAA. *The proof system is complete iff every consistent set is satisfiable.*

Proof Assume that the logic is complete. Let Γ be unsatisfiable, so $\Gamma \models \bot$. By completeness, $\Gamma \vdash \bot$. Thus, Γ is inconsistent.

Assume that every consistent set in the logic is satisfiable Suppose that $\Gamma \models \varphi$. Let $\overline{\varphi}$ be a semantic negation of φ. Then $\Gamma \cup \{\overline{\varphi}\}$ has no models. By our assumption, $\Gamma \cup \{\overline{\varphi}\}$ is inconsistent. Using RAA, we see that $\Gamma \vdash \varphi$. Thus the logic is complete. $\dashv$

Remark Lemmas 6.3.1, 6.3.2, and 6.3.3 have nothing to do with the particular proof system, and indeed they hold for all logical systems which have (RAA). So versions of these results hold for all of the logical systems in the rest of the book. check this

The logic is easily seen to be sound: if $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$. In the next section, we'll see a proof of the completeness of this system.

Theorem 6.3.4 *The logic in Figure 6.2 is sound and complete for $\mathcal{RCA}$.*

Proof We omit the soundness verification. To prove completeness, we use Lemma 6.3.3. We need only show that every theory $\Gamma \subseteq \mathcal{RCA}$ which is consistent in the logic is satisfiable. Also, by Lemma 6.3.2, we may assume that Γ is maximal consistent.

We shall construct a structure $\mathcal{M}$ and prove that it satisfies Γ. First, let $\mathbf{C}^+$ be the set of positive set terms. Then we define $\mathcal{M}$ by:

$$
\begin{array}{lcl}
M & = & \{\langle c_1, c_2, Q\rangle \in \mathbf{C}^+ \times \mathbf{C}^+ \times \{\forall, \exists\} : \Gamma \vdash \exists(c_1, c_2)\} \\
[\![p]\!] & = & \{\langle c_1, c_2, Q\rangle \in M : \Gamma \vdash \forall(c_1, p) \text{ or } \Gamma \vdash \forall(c_2, p)\} \\
\langle c_1, c_2, Q_1\rangle [\![r]\!] \langle d_1, d_2, Q_2\rangle & \text{iff} & \text{either (a) for some } i,\ j, \text{ and } q \in \mathbf{P}, \\
 & & \Gamma \vdash \forall(c_i, \forall(q, r)) \text{ and } \Gamma \vdash \forall(d_j, q); \\
 & & \text{or else (b) } Q_2 = \exists, \text{ and for some } i \text{ and } q \in \mathbf{P}, \\
 & & d_1 = d_2 = q, \text{ and } \Gamma \vdash \forall(c_i, \exists(q, r)).
\end{array}
$$

Note that the set M is non-empty. For let $p \in \mathbf{P}$. If $\Gamma \vdash \exists(p, p)$, then $\langle p, p, \forall\rangle \in M$. Otherwise, $\Gamma \vdash \forall(p, \overline{p})$, and so for all binary atoms r, $\Gamma \vdash \exists(\forall(p, r), \forall(p, r))$ by (W). Thus $\langle c, c, \forall\rangle \in M$, where c is $\forall(p, r)$.

Lemma 6.3.5 *For all $c \in \mathbf{C}^+$,*

$$[\![c]\!] = \{\langle d_1, d_2, Q\rangle \in M : \textit{either } \Gamma \vdash \forall(d_1, c), \textit{ or } \Gamma \vdash \forall(d_2, c)\}.$$

Proof The result for c a unary atom is immediate. We often shall use the resulting fact that if $\Gamma \vdash \exists(p,p)$, then $[\![p]\!] \neq \emptyset$; it contains both $\langle p, p, \forall\rangle$ and $\langle p, p, \exists\rangle$. The main work concerns set terms of the form $\forall(p, r)$ and $\exists(p, r)$. We remark that all set terms referred to in this proof are *positive.*

We begin with $c = \forall(p, r)$. Let $\langle d_1, d_2, Q\rangle \in [\![\forall(p, r)]\!]$. Now, either $\Gamma \vdash \exists(p,p)$ or $\Gamma \nvdash \exists(p,p)$. If the former, then $\langle p, p, \forall\rangle \in [\![p]\!]$. By the semantics of our fragment, $\langle d_1, d_2, Q\rangle [\![r]\!] \langle p, p, \forall\rangle$. By the structure of $\mathcal{M}$, there are i and q giving the derivation from Γ as in the tree on the left below:

$$\dfrac{\begin{matrix}\vdots \\ \forall(d_i, \forall(q, r))\end{matrix} \qquad \dfrac{\begin{matrix}\vdots \\ \forall(p, q)\end{matrix}}{\forall(\forall(q, r), \forall(p, r))}\,(\mathrm{J})}{\forall(d_i, \forall(p, r))}\,(\mathrm{B}) \qquad\qquad \dfrac{\begin{matrix}\vdots \\ \forall(p, \bar{p})\end{matrix}}{\forall(d_j, \forall(p, r))}\,(\mathrm{Z}).$$

This shows that $\Gamma \vdash \forall(d_i, \forall(p, r))$. On the other hand, if $\Gamma \nvdash \exists(p,p)$, we use the assumption that Γ is complete to assert that $\Gamma \vdash \forall(p, \bar{p})$. And then we have the derivation from Γ on the right above, for both j.

Conversely, fix i and suppose that $\Gamma \vdash \forall(d_i, \forall(p, r))$. We claim that $\langle d_1, d_2, Q\rangle$ belongs to $[\![\forall(p, r)]\!]$. For this, take any $\langle b_1, b_2, Q'\rangle \in [\![p]\!]$ so that $\Gamma \vdash \forall(b_j, p)$ for some j. (We are thus using b_1 and b_2 to range over positive set terms, just as the c's and d's do.) Then p, i and j show that $\langle d_1, d_2, Q\rangle [\![r]\!] \langle b_1, b_2, Q'\rangle$. This for all elements of $[\![p]\!]$ shows that $\langle d_1, d_2, Q\rangle \in [\![\forall(p, r)]\!]$.

We next prove the statement of our lemma for $c = \exists(p, r)$.

Let $\langle d_1, d_2, Q\rangle \in [\![\exists(p, r)]\!]$. Thus we have $\langle d_1, d_2, Q\rangle [\![r]\!] \langle b_1, b_2, Q'\rangle$ for some $\langle b_1, b_2, Q'\rangle \in [\![p]\!]$. We first consider case (a) in the definition of our structure $\mathcal{M}$: there are i, j, and q so that $\Gamma \vdash \forall(d_i, \forall(q, r))$ and $\Gamma \vdash \forall(b_j, q)$. We have $\Gamma \vdash \exists(b_1, b_2)$, since $\langle b_1, b_2, Q'\rangle \in M$. Further, let k be such that $\Gamma \vdash \forall(b_k, p)$. We show the desired conclusion using a derivation from Γ:

$$\dfrac{\begin{matrix}\vdots \\ \forall(d_i, \forall(q, r))\end{matrix} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots \\ \exists(b_1, b_2)\end{matrix}}{\exists(b_k, b_j)} \qquad \begin{matrix}\vdots \\ \forall(b_j, q)\end{matrix}}{\exists(b_k, q)}\,(\mathrm{D1}) \qquad \begin{matrix}\vdots \\ \forall(b_k, p)\end{matrix}}{\exists(q, p)}\,(\mathrm{D1})}{\forall(\forall(q, r), \exists(p, r))}\,(\mathrm{L})}{\forall(d_i, \exists(p, r))}\,(\mathrm{B})$$

This concludes the work in case (a). In case (b), $Q' = \exists$, there is some $q \in \mathbf{P}$ such that $b_1 = b_2 = q$, and for some i, $\Gamma \vdash \forall(d_i, \exists(q, r))$. Again we have $\Gamma \vdash \forall(q, p)$. So we have a derivation from Γ as follows:

$$\dfrac{\begin{array}{c}\vdots\\ \forall(d_i,\exists(q,r))\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ \forall(q,p)\end{array}}{\forall(\exists(q,r),\exists(p,r))}\,(\mathrm{K})}{\forall(d_i,\exists(p,r))}\,(\mathrm{B})$$

At this point, we know that if $\langle d_1, d_2, Q\rangle \in [\![\exists(p,r)]\!]$, then $\Gamma \vdash \forall(d_i,\exists(p,r))$ for some i. We now verify the converse. Let $\langle d_1, d_2, Q\rangle \in M$, and fix i such that $\Gamma \vdash \forall(d_i,\exists(p,r))$. Then $\Gamma \vdash \exists(d_1,d_2)$. We thus have a derivation from Γ:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \exists(d_1,d_2)\end{array}}{\exists(d_i,d_i)}\,(\mathrm{I}) \qquad \begin{array}{c}\vdots\\ \forall(d_i,\exists(p,r))\end{array}}{\exists(d_i,\exists(p,r))}\,(\mathrm{D1})}{\exists(p,p)}\,(\mathrm{II})$$

This goes to show that $\langle p,p,\exists\rangle \in M$. By the construction of $\mathcal{M}$, $\langle d_1,d_2,Q\rangle[\![r]\!]\langle p,p,\exists\rangle$, and $\langle p,p,\exists\rangle \in p^{\mathcal{M}}$. So $\langle d_1,d_2,Q\rangle \in [\![\exists(p,r)]\!]$. This completes the proof. $\dashv$

Lemma 6.3.6 *The interpretation of each adjective atom a is transitive.*

Proof Assume that

$$\langle b_1,b_2,Q_1\rangle\ [\![a]\!]\ \langle c_1,c_2,Q_2\rangle\ [\![a]\!]\ \langle d_1,d_2,Q_3\rangle.$$

We shall use several times the fact that since $\langle c_1,c_2,Q_2\rangle$ belongs to the model, $\Gamma \vdash \exists(c_1,c_2)$. We have four cases.

Case 1: for some i, j, k, l, and $q_1,q_2 \in \mathbf{P}$, $\Gamma \vdash \forall(b_i,\forall(q_1,a))$, $\Gamma \vdash \forall(c_j,q_1)$, $\Gamma \vdash \forall(c_k,\forall(q_2,a))$, and $\Gamma \vdash \forall(d_l,\forall(q_2,a))$. We have $\Gamma \vdash \exists(c_j,c_k)$ also. Now we have derivation from Γ:

$$\dfrac{\begin{array}{c}\vdots\\ \forall(b_i,\forall(q_1,a))\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \forall(c_k,\forall(q_2,a))\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ \forall(c_j,q_1)\end{array} \quad \begin{array}{c}\vdots\\ \exists(c_j,c_k)\end{array}}{\exists(q_1,c_k)}}{\exists(q_1,\forall(q_2,a))}}{\forall(\forall(q_1,a),\forall(q_2,a))}\,(\mathrm{tr3})}{\forall(b_i,\forall(q_2,a))}\,(\mathrm{B})$$

And now we see that $\langle b_1,b_2,Q_1\rangle[\![a]\!]\langle d_1,d_2,Q_3\rangle$, using alternative (a) in the definition of $[\![r]\!]$.

Case 2: for some i, j, and $q_1 \in \mathbf{P}$, $\Gamma \vdash \forall(b_i,\forall(q_1,a))$ and $\Gamma \vdash \forall(c_j,q_1)$; $Q_3 = \exists$, and for some k and $q_2 \in \mathbf{P}$, $d_1 = d_2 = q_2$, and $\Gamma \vdash \forall(c_k,\exists(q_2,a))$. This time, we have $\Gamma \vdash \exists(q_1,c_k)$, and by a derivation similar to what we saw in Case 1, $\Gamma \vdash \forall(b_i,\exists(q_2,a))$. In this case, alternative (b) shows that $\langle b_1,b_2,Q_1\rangle[\![a]\!]\langle d_1,d_2,Q_3\rangle$.

Case 3: $Q_2 = \exists$, *and for some* i *and* $q_1 \in \mathbf{P}$, $c_1 = c_2 = q_1$, *and* $\Gamma \vdash \forall(b_i, \exists(q_1, a))$; *and also for some* j, k, *and* $q_2 \in \mathbf{P}$, $\Gamma \vdash \forall(c_j, \forall(q_2, a))$ *and* $\Gamma \vdash \forall(d_k, q_2)$. Thus c_j at the end is the same as q_1. The proof system now shows that $\Gamma \vdash \forall(b_i, \forall(q_2, a))$. We again finish the case with an application of (a).

Case 4: $Q_2 = \exists$, *and for some* i *and* $q_1 \in \mathbf{P}$, $c_1 = c_2 = q_1$, *and* $\Gamma \vdash \forall(b_i, \exists(q_1, a))$; *and also* $Q_3 = \exists$, *and for some* j *and* $q_2 \in \mathbf{P}$, $d_1 = d_2 = q_2$, *and* $\Gamma \vdash \forall(c_l, \exists(q_2, a))$. This time we see that $\Gamma \vdash \forall(b_i, \exists(q_2, a))$, and we use (b) to see that $\langle b_1, b_2, Q_1\rangle [\![a]\!] \langle d_1, d_2, Q_3\rangle$.

This completes the proof. ⊣

Lemma 6.3.7 $\mathcal{M} \models \Gamma$.

Proof The proof is by cases on the various sentence types in the current logic. Using the fact that formulas $\exists(e, f)$ and $\exists(f, e)$ are interderivable in the logic, and similarly for $\forall(\bar{e}, \bar{f})$ and $\forall(f, e)$, we may take all sentences to have one of the forms:

$$\forall(c^+, d^+), \quad \forall(c^+, \overline{d^+}), \quad \exists(c^+, d^+), \quad \exists(c^+, \overline{d^+}),$$

where c^+ and d^+ range over *positive* set terms. In the remainder of the proof, we omit the $^+$-superscripts for clarity: i.e. c and d range over positive set terms.

Let $\varphi \in \Gamma$ be $\forall(c, d)$. Using (B) and Lemma 6.3.5, we see that $[\![c]\!] \subseteq [\![d]\!]$.

Let $\varphi \in \Gamma$ be $\forall(c, \bar{d})$. Suppose towards a contradiction that $\mathcal{M} \not\models \varphi$. Let $\langle b_1, b_2, Q\rangle \in [\![c]\!] \cap [\![d]\!]$. Let i and j be such that $\Gamma \vdash \forall(b_i, c)$ and $\Gamma \vdash \forall(b_j, d)$. Then using (B), $\Gamma \vdash \forall(b_i, \bar{d})$. And since $\Gamma \vdash \exists(b_i, b_j)$, we use (D1) to see that $\Gamma \vdash \exists(d, \bar{d})$. So Γ is inconsistent, a contradiction.

If $\varphi \in \Gamma$ is $\exists(c, d)$, then $(c, d, \exists) \in M$. Indeed, $(c, d, \exists) \in [\![c]\!] \cap [\![d]\!]$, by Rule (T) and Lemma 6.3.5.

Finally, consider the case when $\varphi \in \Gamma$ is of the form $\exists(c, \overline{d})$. Then, using (I), $\Gamma \vdash \exists(c, c)$, so $\langle c, c, \forall\rangle \in M$. Suppose towards a contradiction that $\mathcal{M} \models \forall(c, d)$. Then $\langle c, c, \forall\rangle \in [\![d]\!]$. But then we have $\Gamma \vdash \forall(c, d)$, by Lemma 6.3.5 again. One application of (D2) now shows that $\Gamma \vdash \exists(d, \overline{d})$. Thus we have a contradiction to the consistency of Γ. ⊣

This completes the proof of Theorem 6.3.4. ⊣

A last point: extensions of the system It is possible to go somewhat further in this direction. We could add proof rules for the irreflexivity of comparative adjective phrases, and for matter we can also force the domains to be finite (or to be infinite). We can also add rules for the converse relations, thus relating bigger than and smaller than. (Technically, this last addition is harder to handle: it leads to a rather large set of axioms.) We still would have complete and decidable logical systems. We can also add a few more features to syllogistic logics like this.

Sources Most of the material in this chapter is from Moss [?], Pratt-Hartmann and Moss [?].

To the best of my knowledge, the first presentation of a complete proof-system for a fragment close to the relational syllogistic seems to be Nishihara, Morita, and Iwata [?]. This logic is in effect a relational version of Łukasiewicz', in that formulas roughly similar to those of $\mathcal{R}$ are treated as atoms of a propositional calculus. The authors provide axiom-schemata which, together with the usual axioms of propositional logic, yield a complete proof-system for the language in question. Actually, the propositional atoms in this language are allowed to feature n-ary predicates for all $n \geq 1$. However, the rather strange restrictions on quantifier-scope (existentials must always outscope universals), mean that this language is primarily of interest for atoms featuring only unary and binary predicates; these atoms (and their negations) then essentially correspond the formulas of our fragment $\mathcal{R}$.

6.4 Exercises

Exercise 32. Formalize Example 5.1 in $\mathcal{RCA}$ and then construct a derivation for your formalization using the rules of this section.

Exercise 33. Consider the following inference:

Every giraffe is taller than every gnu
Some gnu is taller than every lion
<u>Some lion is taller than some zebra</u>
Every giraffe is taller than some zebra

The inference is valid. Construct a derivation in the system.

Exercise 34. Let Γ consist of the formalizations of the following three sentences:

Some giraffe is a female animal
Every giraffe is taller than every female animal
Every female animal is not taller than some female animal

In the last sentence, we use the wide-scope reading; so for every female animal x is not taller there is some female animal y (depending on x) so that x is not taller than y.

(i) Show that Γ is unsatisfiable.

(ii) Show that Γ is inconsistent in the logic.

Exercise 35. Show the following principles are all derivable in our logic for $\mathcal{RCA}$:

$$\frac{\forall(p, \forall(q', t)) \qquad \exists(q, q')}{\forall(p, \exists(q, t))}\ (\forall\forall) \qquad \frac{\exists(p, \exists(q, t)) \qquad \forall(q, q')}{\exists(p, \exists(q', t))}\ (\exists\exists)$$

$$\frac{\forall(p, \exists(q, t)) \qquad \forall(q, q')}{\forall(p, \exists(q', t))}\ (\forall\exists) \qquad \frac{\forall(q, \bar{c}) \qquad \exists(p, c)}{\exists(p, \bar{q})}\ (\mathrm{D3})$$

$[\![\mathsf{men}]\!] = \{a, d, f\}$

$[\![\mathsf{women}]\!] = \{b, c, e\}$

$[\![\mathsf{walks}]\!] = \{a, b, c\}$

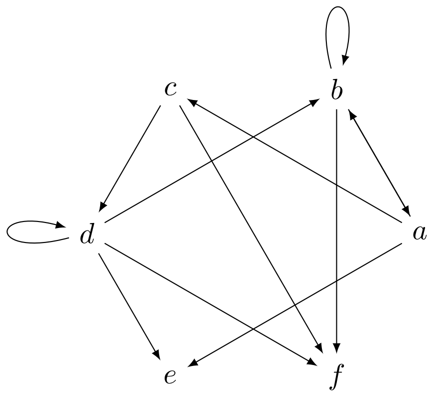


Figure 6.3: The model $\mathcal{M}$ from Exercise 37 with the interpretation $[\![\mathsf{see}]\!]$ drawn as a *graph* rather than listed as a set of ordered pairs. We also take $[\![\mathsf{taller}]\!]$ to be the strict alphabetic order relation on M.

Exercise 36. The language $\mathcal{R}$ is smaller than $\mathcal{RCA}$ because it doesn't have nested set terms. It consists of the sentences shown in Figure **??**. Prove that we get a sound and complete logic for $\mathcal{R}$ using some of the rules in Figure 6.2: (T), (B), (D1), (D2), (I); together with (A) from Example 6.7; together with the rules D3), (∀∀)), (∃∃), and (∀∀) from Exercise 35 above.

Exercise 37. In Figure 6.3, you'll see a model for this fragment. Technically, $M = \{a, \ldots, f\}$,

$$[\![\mathsf{see}]\!] \quad = \quad \{(a, b), (a, c), (a, e), (c, d), (b, a), (b, b), (b, f), (c, f), (d, b), (d, d), (d, e), (d, f)\},$$

and $[\![\mathsf{bigger}]\!] = \{(b, a), (c, a), \ldots, (f, e)\}$.

Find the following sets:

(i) $[\![\overline{\mathsf{man}}]\!]$

(ii) $[\![\overline{\mathsf{see}}]\!]$

(iii) $[\![\exists(\mathsf{walks}, \mathsf{see})]\!]$

(iv) $[\![\forall(\overline{\mathsf{walks}}, \mathsf{see})]\!]$

(v) $[\![\exists(\forall(\overline{\mathsf{walks}}, \mathsf{see}), \mathsf{bigger})]\!]$

Exercise 38. This is a continuation of Exercise 37. Translate the following sentences from English into our formalism and see if they are true or false in the model $\mathcal{M}$ in Figure 6.3:

(i) Every man walks.

(ii) Some men walk.

(iii) Some men see every woman.

(iv) Every man sees some woman who is taller than some man.

(v) Everyone who sees every man doesn't see someone who is taller than some woman.

(vi) Everyone who sees every man doesn't see anyone who is taller than some woman.

Exercise 39. A relation R on a set X is *reflexive* if for all $x \in M$, x is related to itself by R. R is *irreflexive* if for all $x \in M$, x is not related to itself by R.

(i) Let $\mathcal{M}$ be a model in which the interpretation $[\![a]\!]$ of every adjective atom is *irreflexive* Show that for all set terms c and all adjectives a, $\mathcal{M} \models \forall(c, \exists(c, \overline{a}))$.

(ii) Let Γ be a maximal consistent theory in the logic for $\mathcal{RCA}$, and let $\mathcal{M} = \mathcal{M}(\Gamma)$ be the model constructed in Section 6.3. Which points in M are related to themselves by the interpretations of adjectives?

(iii) Continuing with the last part, suppose that we add all sentences of the form

$$\frac{}{\forall(c, \exists(c, \overline{a}))}\ (\textsc{irr}) \tag{6.2}$$

as axioms to the logic. Suppose again that Γ is maximal consistent in the new logic, and that we construct $\mathcal{M}(\Gamma)$ as before. Which points in M are related to themselves by the interpretations of adjectives? (Exercise 34 is relevant here.)

(iv) Modify $\mathcal{M}$ to get an irreflexive model $\mathcal{N}$ of Γ. This proves that the logic obtained by adding the new axioms $\forall(c, \exists(c, \overline{a}))$ is a sound and complete logic when we restrict the interpretations of the adjectives to be irreflexive relations.

[Hint: The idea is to take each reflexive point p in M that you found in part 3 and to replace p with infinitely many copies $p_1, p_2, \ldots$. We want to have $p_i [\![a]\!] p_j$ iff $i < j$. And we want to have the rest of the structure of $\mathcal{N}$ be as close to $\mathcal{M}$ as possible. This exercise requires a fair amount of checking; most of the work is similar to what we saw in the proof of Theorem 6.3.4.]

Exercise 40. We continue to study models $\mathcal{M}$ with the property that the interpretation of the comparative adjectives are required to be irreflexive (and of course transitive). We also want to restrict attention to models $\mathcal{M}$ whose domain M is *finite*.

(i) Show that the following rule of inference is sound for this class:

$$\frac{\exists(c, c,)}{\exists(c, \forall(c, \overline{a}))}\ (\textsc{fin})$$

(ii) Show that the logic of $\mathcal{RCA}$ together with the rules ($\textsc{Irr}$) from (6.2) and ($\textsc{fin}$) are complete for the class of finite models which interpret the adjectives by irreflexive (and transitive) relations.

7 $\mathcal{S}^\dagger$(card): Reasoning about the Sizes of Sets

Up until now, all of our logical systems were fragments of first-order logic. And for the most part, many people have assumed that if a fragment of natural language was to be formalized at all, it would wind up being a fragment of first-order logic. This chapter shows that this is not true: we'll study a logical system which is provably not first-order.

7.1 $\mathcal{S}^\dagger$(card): syllogistic logic with negation and cardinality comparison

The logic $\mathcal{S}^\dagger$(card) merges two logical systems which we have aready seen. The language of this fragment has nouns p, q, $\ldots$, as we have seen previously. It also has negated nouns p', q, $\ldots$. In addition to the sentences $\forall(p,q)$ and $\exists(p,q)$, it has sentences $\exists^{\geq}(p,q)$ and $\exists^{>}(p,q)$. The semantics begins with a universe M and interpretations $[\![p]\!]$ for all nouns p. We assume that in the syntax, $p'' = p$ for all p. Thus, the semantics of each complemented noun p' is $M \setminus [\![p]\!]$. The sentences $\exists^{\geq}(p,q)$ and $\exists^{>}(p,q)$ have the following semantics:

$$\begin{array}{lll} \mathcal{M} \models \exists^{\geq}(p,q) & \text{iff} & |[\![p]\!]| \geq |[\![q]\!]| \\ \mathcal{M} \models \exists^{>}(p,q) & \text{iff} & |[\![p]\!]| > |[\![q]\!]| \end{array}$$

We read "$\exists^{\geq}(p,q)$" as "there are at least as many p as q", and we read "$\exists^{>}(p,q)$" as "there are more p than q." So $\exists^{\geq}(p,p')$ might be read as "the p's are at least half of the objects in the universe." Similarly, $\exists^{\geq}(p,p')$ might be read as "the p's are at most half of the objects in the universe." We can also read $\exists^{>}(p,p')$ might be read as "the p's are more than half of the objects in the universe," and $\exists^{>}(p',p)$ as "the p's are less than half of the objects in the universe,"

We are interested in working with this semantics only on *finite universes*. This is because the logic is stronger this way. That is, some of the rules which we shall see shortly are not sound for infinite universes.

The rules of the system are listed in Figure 7.1. One rule which uses the finiteness assertion is (Card Mix). It says that if all y are x, and there are at least as many elements in the bigger set y as in x, then the sets have to be the same.

We turn to the rules at the bottom of Figure 7.1, since they show the interaction of the different sentence types and also involve the "half" interpretation from above.

The logic has two *Ex falso quodlibet rules*, listed at the bottom of Figure 7.1. However, the second is derivable from the first.

$$\frac{}{\forall(p,p)}\ \text{(axiom)} \qquad \frac{\forall(n,p) \quad \forall(p,q)}{\forall(n,q)}\ \text{(Barbara)} \qquad \frac{\exists(p,q)}{\exists(p,p)}\ \text{(some)}$$

$$\frac{\exists(q,p)}{\exists(p,q)}\ \text{(conversion)} \qquad \frac{\forall(p,q)}{\forall(q',p')}\ \text{(anti)} \qquad \frac{\forall(p,p')}{\forall(p,q)}\ \text{(zero)}$$

$$\frac{\exists(p,n) \quad \forall(n,q)}{\exists(p,q)}\ \text{(Darii)} \qquad \frac{\forall(p',p)}{\forall(q,p)}\ \text{(one)} \qquad \frac{\forall(p,q)}{\exists^{\geq}(q,p)}\ \text{(subset-size)}$$

$$\frac{\exists^{\geq}(p,q)}{\exists^{\geq}(q',p')}\ \text{(card-mon)} \qquad \frac{\exists^{\geq}(p,q)}{\exists^{\geq}(q',p')}\ \text{(card-anti)} \qquad \frac{\forall(p,q) \quad \exists^{\geq}(p,q)}{\forall(q,p)}\ \text{(card-mix)}$$

$$\frac{\exists(p,p) \quad \exists^{\geq}(p,q)}{\exists(q,q)}\ \text{(card-}\exists\text{)} \qquad \frac{\forall(q,p) \quad \exists(p,q')}{\exists^{>}(p,q)}\ \text{(more)} \qquad \frac{\exists^{>}(p,q)}{\exists(p,q')}\ \text{(more-some)}$$

$$\frac{\exists^{>}(p,q)}{\exists^{\geq}(p,q)}\ \text{(more-at least)} \qquad \frac{\exists^{>}(n,p) \quad \exists^{\geq}(p,q)}{\exists^{>}(n,q)}\ \text{(more-left)} \qquad \frac{\exists^{>}(q,p)}{\exists^{>}(p',q')}\ \text{(more-anti)}$$

$$\frac{\exists(p,p) \quad \exists^{\geq}(q,q')}{\exists(q,q)}\ \text{(int)} \qquad \frac{\exists^{\geq}(p,p') \quad \exists^{\geq}(q',q)}{\exists^{\geq}(p,q)}\ \text{(half)} \qquad \frac{\exists^{>}(p,p') \quad \exists^{\geq}(q',q)}{\exists^{>}(p,q)}\ \text{(strict half)}$$

$$\frac{\exists^{\geq}(p,p') \quad \exists^{\geq}(q,q') \quad \exists(p',q')}{\exists(p,q)}\ \text{(maj)}$$

$$\frac{\exists(p,q) \quad \forall(q,q')}{\varphi}\ \text{(X)} \qquad \frac{\exists^{>}(p,q) \quad \exists^{\geq}(q,p)}{\varphi}\ \text{(X)}$$

Figure 7.1: Rules for $\mathcal{S}^\dagger$(card). **(card-mon) and (card-anti) are the same!**

Example 7.1 *All x are non-y* follows from the list of assumptions below:

(i) *There are at least as many non-y as y*

(ii) *There are at least as many non-z as z*

(iii) *All x are z*

(iv) *All non-y are z*

Here is a formal proof in our system:

$$\dfrac{\forall(x,z) \qquad \dfrac{\forall(\overline{y},z) \qquad \dfrac{\exists^{\geq}(\overline{y},y) \quad \exists^{\geq}(\overline{z},z)}{\exists^{\geq}(\overline{y},z)}\ (\textsc{Half})}{\forall(z,\overline{y})}\ (\textsc{Card Mix})}{\forall(x,\overline{y})}\ (\textsc{Barbara})$$

We write $\exists^{\geq}(x,y)$ for *There are at least as many x as y*, and we are interested in adding these sentences to our fragments. We are usually interested in sentences in this fragment on *finite* models. We write $|S|$ for the cardinality of the set φ. The semantics is that $\mathcal{M} \models \exists^{\geq}(x,y)$ iff $|[\![x]\!]| \geq |[\![y]\!]|$ in $\mathcal{M}$.

Remark In the remainder of this chapter Γ denotes a *finite* set of sentences. The reason is that the logic is not so well-behaved for infinite sets of assumptions; see Exercise 41.

We need a little notation at this point.

Definition Let Γ be a (finite) set of sentences. As before, we write $x \leq y$ for $\Gamma \vdash$ All x are y. Note that Γ is left off the notation. And we write $x \equiv y$ for $x \leq y \leq x$.

We write $x \leq_c y$ for $\Gamma \vdash \exists^{\geq}(y,x)$. We also write $x \equiv_c y$ for $x \leq_c y \leq_c x$, and $x <_c y$ for $x \leq_c y$ but $x \not\equiv_c y$.

Finally, we write $x <_{more} y$ if $\Gamma \vdash \exists^{>}(y,x)$.

Proposition 7.1.1 *Let $\Gamma \subseteq \mathcal{L}(\mathsf{all}, \exists^{\geq})$ be a (finite) set. Let $\mathcal{V}$ be the set of variables in Γ.*

(i) If $x \leq y$, then $x \leq_c y$.

(ii) $(\mathcal{V}, \leq_c)$ is a preorder: a reflexive and transitive relation.

(iii) If $x \leq_c y \leq x$, then $x \leq y$.

(iv) If $x \leq_c y$, $x \equiv x'$, and $y \equiv y'$, then $x' \leq_c y'$.

(v) $(\mathcal{V}, \leq_c)$ is pre-wellfounded: a preorder with no descending sequences in its strict part.

Proof Part (1) uses the (SUBSET-SIZE) rule. In part (2), the reflexivity of $\leq_c$ comes from that of $\leq$ and part (1); the transitivity is by the second rule of $\exists^{\geq}$. Part (3) is by the last rule of $\exists^{\geq}$. Part 4 uses part (1) and transitivity. Part 5 is just a summary of the previous parts. $\dashv$

Preliminary: listings of finite transitive sets A *listing* of a set is an enumeration without repetitions.

Lemma 7.1.2 *Let $(T, <)$ be a finite set with a transitive, irreflexive relation. Then there is a listing of T as*

$$t_1, t_2, \ldots, t_n$$

with the property that if $t_i < t_j$, then $i < j$. In words, the $<$-predecessors of each point are listed before it.

Proof By induction on the size of T. If T has 0 or 1 element, the result is trivial. Assume the result for orders of size n, and let $(T, \leq)$ be of size $n+1$. Let x be such that there is no $y < x$. Such x must exist since T is finite. (Here is the argument in more detail: Suppose towards a contradiction that for every z there were some $w < z$, we would have an infinite sequence $z_0 > z_1 > \cdots > z_n > \cdots$. By finiteness there is $m < n$ so that $z_m = z_n$. But by transitivity we have $z_m > z_n$. And this contradicts the irreflexivity of $<$.) Let $T' = T \setminus \{x\}$, and consider T' with the restriction $<'$ of $<$. This order $(T', <')$ is again transitive and irreflexive, and it has size n. By induction hypothesis, there exists a listing of T' as $t_1, t_2, \ldots, t_n$. Then we take for the listing on T the list $x, t_1, t_2, \ldots, t_n$. $\dashv$

We also need the following refinement of Lemma 7.1.2.

Lemma 7.1.3 *Let $(T, <)$ be a finite set with a transitive, irreflexive relation. Suppose that $x \not\leq y$ in T. Then there is a listing of T as*

$$t_1, t_2, \ldots, t_n$$

with the following properties:

(i) If $t_i < t_j$, then $i < j$: the $<$-predecessors of each point are listed before it.

(ii) If $t_i = x$ and $t_j = y$, then $j < i$. That is, x comes after y in the listing.

Proof Let $t_1, t_2, \ldots, t_n$ be an enumeration as in Lemma 7.1.2. If y comes before x in the enumeration, then we are done. Otherwise, x comes before y in the enumeration. Let

$$S = \{z : z \text{ is between } x \text{ and } y \text{ (inclusive) in the enumeration, and } x \not\leq z\}$$

Notice that $y \in S$. Move the points in S to just before x, in their order in the original enumeration. We need to check that the $<$-predecessors of each point are listed before

$$\frac{}{\forall(p,p)}\ (\text{axiom}) \qquad \frac{\forall(n,p) \quad \forall(p,q)}{\forall(n,q)}\ (\text{Barbara}) \qquad \frac{\exists(p,q)}{\exists(p,p)}\ (\text{some})$$

$$\frac{\exists(q,p)}{\exists(p,q)}\ (\text{conversion}) \qquad \frac{\exists(p,n) \quad \forall(n,q)}{\exists(p,q)}\ (\text{Darii}) \qquad \frac{\forall(p,q)}{\exists^{\geq}(q,p)}\ (\text{subset-size})$$

$$\frac{\exists^{>}(p,q)}{\exists^{\geq}(p,q)}\ (\text{more-at least}) \qquad \frac{\exists(p,p) \quad \exists^{\geq}(p,q)}{\exists(q,q)}\ (\text{card-}\exists) \qquad \frac{\forall(p,q) \quad \exists^{\geq}(p,q)}{\forall(q,p)}\ (\text{card-mix})$$

$$\frac{\exists^{>}(n,p) \quad \exists^{\geq}(p,q)}{\exists^{>}(n,q)}\ (\text{more-left})$$

Figure 7.2: Rules for $\mathcal{S}$(card).

it. The only points to worry about are those in S, and for some $z \in S$, we only need to show that z has no $<$-predecessors which are $\geq x$ are also in S. Suppose that towards a contradiction that w were such a predecessor. Then $w < z \leq y$ and, since $w \notin S$, we have $x \leq w$. Thus $x \leq w < z$, and this contradicts $z \in S$. $\dashv$

7.2 $\mathcal{S}$(card) and the Construction Lemma

The Completeness Theorem for $\mathcal{S}^{\dagger}$(card) takes a fair amount of work, and it makes sense to study a smaller system first. We are going to study a logic $\mathcal{S}$(card) as a stepping stone towards $\mathcal{S}^{\dagger}$(card). This smaller language $\mathcal{S}$(card) has everything $\mathcal{S}^{\dagger}$(card) has, *but not complemented variables.*

At this point, we need to emphasize that there is a difference between $<_c$ and $<_{more}$. When we write $a <_c b$, we mean that $\Gamma \vdash$ At least as many b as a and $\Gamma \not\vdash$ At least as many a as b. This is weaker than $\Gamma \vdash$ More b than a. If Γ is consistent and $\Gamma \vdash$ More b than a, then $\Gamma \vdash$ At least as many b as a. But the converse does not hold.

Lemma 7.2.1 *Let Γ be a finite set of sentences in $\mathcal{S}$(card) which is consistent in the logic. Let $\mathcal{V}$ be a finite set of variables which include all the variables occurring in Γ. Let $\mathcal{V}/\equiv_c$ be the set of equivalence classes of variables in Γ under $\equiv_c$. Let*

$$[u_0], [u_2], \ldots, [u_k]$$

be a listing of $\mathcal{V}/\equiv_c$ with the property that if $u_i <_c u_j$, then $i < j$.

Then there is a model $\mathcal{M} = \mathcal{M}_\Gamma$ such that for all $a, b \in \mathcal{V}$ and $0 \leq i, j \leq k$,

(α) If $a \leq b$, then $[\![a]\!] \subseteq [\![b]\!]$.

(β) If $i < j$ *and* $\exists^{\geq}(u_j, u_i)$, *then* $|[\![u_i]\!]| \leq |[\![u_j]\!]|$.

(γ) If $i < j$ *and* $\exists^{>}(u_j, u_i)$, *then* $|[\![u_i]\!]| < |[\![u_j]\!]|$.

Moreover, $\mathcal{M} \models \Gamma$.

Proof We define by recursion on $i \leq k$ the interpretation $[\![v]\!]$ of all $v \in [u_i]$. Suppose that for all $j < i$ and all $w \equiv_c u_j$, we have an interpretation $[\![w]\!]$. For $v \in [u_i]$, let

$$\begin{array}{rcl} B_v & = & \bigcup\{[\![x]\!] : x \leq v \text{ and } (\exists j < i)(x \equiv_c u_j)\} \\ C_v & = & \bigcup\{\varphi \in \Gamma : \varphi \text{ is } \exists(v,u) \text{ or } \exists(u,v) \text{ for some } u\} \\ D_v & = & B_v \cup C_v \end{array}$$

We start by setting $[\![v]\!]$ to be D_v, but we might need to add more points in order to satisfy some of the requirements. Let

$$n \quad = \quad \max\{|D_v| : v \in [u_i]\}.$$

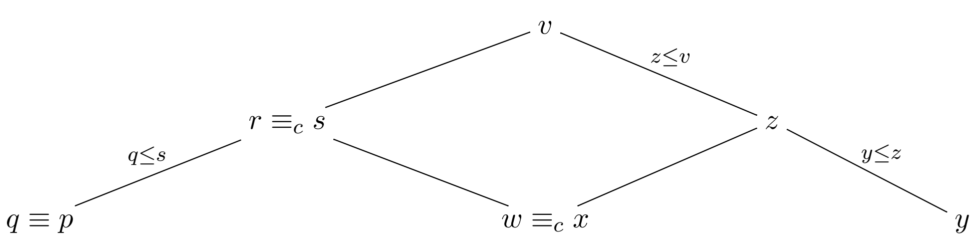
For each v, add fresh points to $[\![v]\!]$ in order that they all have the same size. That is, add $n - |D_v|$ points to $[\![v]\!]$. Finally, if there is some $i < j$ such that $\exists^{>}(u_j, u_i)$, and yet $|[\![u_j]\!]| = |[\![u_i]\!]|$, then add one fresh point to $[\![v]\!]$ for all $v \in [u_i]$.

It is not hard to see that (α), (β), and (γ) all hold. Further, we check that $\mathcal{M} \models \Gamma$. $\dashv$

Example Suppose that Γ is the following set of sentences:

All p are q	There are at least as many x as w
There are at least as many q as p	There are at least as many x as r
All q are s	All y are z
There are at least as many r as s	There are at least as many w as z
There are at least as many s as r	All z are v
There are at least as many w as x	There are at least as many s as v

Here is a picture of the relations $\leq$ and $\leq_c$. The lines are the $\leq_c$ relation, reading upward, with the stronger $\leq$ relations shown.



We start with distinct elements $*_p = *_q$, $*_r$, $*_s$, $*_w$, $*_x$, $*_y$, $*_z$.
Let's list $\mathcal{V}/\equiv_c$ as $[p]$, $[w]$, $[r]$, $[y]$, $[x]$, $[v]$.

We follow the proof of Lemma 7.2.1 using this listing. Each time we need fresh elements, we shall use numbers.

$$\begin{array}{lcl@{\qquad}lcl}
[\![p]\!] & = & \{*_p\} & [\![r]\!] & = & \{*_r, 3, 4, 5, 6, 7\} \\
[\![q]\!] & = & \{*_p\} & [\![s]\!] & = & \{*_p, *_s, 8, 9, 10, 11\} \\
[\![w]\!] & = & \{*_w, 1\} & [\![y]\!] & = & \{*_y, 12, \ldots, 22, 23\} \\
[\![x]\!] & = & \{*_x, 2\} & [\![z]\!] & = & \{*_y, *_z, 12, \ldots, 22, 23\} \\
 & & & [\![v]\!] & = & \{*_v, *_y, *_z, 12, \ldots, 23, 24, 25, \ldots, 39\}
\end{array}$$

Observe that the model satisfies Γ, and that the only inclusion ($\subseteq$) relations between the interpretations of different atoms are the ones implied by Γ: $[\![q]\!] \subseteq [\![s]\!]$, $[\![y]\!] \subseteq [\![z]\!]$, $[\![z]\!] \subseteq [\![v]\!]$, and $[\![y]\!] \subseteq [\![v]\!]$.

7.3 The Completeness Theorem for $\mathcal{S}$(card)

Theorem 7.3.1 *The logic of Figure 7.2 is complete for $\mathcal{S}(card)$.*

Proof We show that for all sentences φ, if $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$. Actually, we assume that Γ is *consistent*, and then we prove that if $\Gamma \not\vdash \varphi$, then there is a model of Γ where φ fails. We break into two cases, depending on what kind of sentence φ is.

The first case: φ is of the form $\exists(x, y)$ In this case, we check by induction on i that

$$[\![x]\!] \cap [\![y]\!] \cap \textstyle\bigcup_{j<i} [\![u_i]\!] \quad = \quad \emptyset.$$

The next case: φ is of the form $\exists^{\geq}(x, y)$ Assuming that $\Gamma \not\vdash \varphi$, we see that $y \not\leq_c x$. We start with a listing of $\mathcal{V}/\equiv_c$ which puts $[x]$ before $[y]$. And then at the step when we define $[\![y]\!]$ (and also $[\![v]\!]$ for all $v \equiv_c y$, we need only add a fresh element to $[\![y]\!]$ (if need be) in order to get a model where $|[\![y]\!]| > |[\![x]\!]|$.

The next case: φ is of the form $\forall(x, y)$ If $x \equiv_c y$, then we also have $\neg(x \equiv y)$. So when we define $[\![x]\!]$ and $[\![y]\!]$, we can add a point to $[\![x]\!]$ which is not in $[\![y]\!]$. If $x <_c y$, then our listing of $\mathcal{V}/\equiv_c$ puts $[x]$ before $[y]$. When we define $[\![x]\!]$, make sure that it is not empty. Then when we define $[\![y]\!]$, our construction has arranged that it not be a superset of $[\![x]\!]$. Finally, we have the case when $y <_c x$. In this case, when we define $[\![x]\!]$ we already have $[\![y]\!]$, and we can add a fresh point to $[\![x]\!]$ to insure that it not be a subset of $[\![x]\!]$.

The final case: φ is of the form $\exists^{>}(x, y)$ If $x \equiv_c y$, then our model construction gives a model of Γ where $[\![x]\!]$ and $[\![y]\!]$ are of the same size, as desired. If $x <_c y$, then our construction gives a model where $|[\![x]\!]| \leq |[\![y]\!]|$. Again, this is what we want. Finally, we have the case when $y <_c x$. This is the trickiest and most interesting case.

This concludes the proof of our theorem. $\dashv$

7.4 The Completeness Theorem for $\mathcal{S}^{\dagger}$(card)

We now use the Construction Lemma 7.2.1 and also Theorem 7.3.1 to prove the completeness of our system for $\mathcal{S}^{\dagger}$(card).

7.4.1 Small, large, and half

Definition Γ is *trivial* if for all p, $\Gamma \vdash \forall(p, \overline{p})$. Otherwise, Γ is *non-trivial.*

Lemma 7.4.1 *Let Γ be consistent and non-trivial. There is a partition of the unary atoms into three sets*

$$\mathsf{small}, \mathsf{half}, \text{ and } \mathsf{large}$$

such that

(i) If $\Gamma \vdash \exists^{>}(\overline{p}, p)$, then $p \in \mathsf{small}$.

(ii) If $\Gamma \vdash \exists^{\geq}(\overline{p}, p)$, then $p \in \mathsf{small}$ or $p \in \mathsf{half}$.

(iii) If $\Gamma \vdash \exists^{\geq}(p, \overline{p})$ and also $\Gamma \vdash \exists^{\geq}(\overline{p}, p)$, then $p, \overline{p} \in \mathsf{half}$.

(iv) If $\Gamma \vdash \exists^{>}(p, \overline{p})$, then $p \in \mathsf{large}$.

(v) If $\Gamma \vdash \exists^{\geq}(p, \overline{p})$, then $p \in \mathsf{large}$ or $p \in \mathsf{half}$.

(vi) If $p \in \mathsf{small}$, then $\overline{p} \in \mathsf{large}$.

(vii) If $p \in \mathsf{half}$, then $\overline{p}$ in half.

(viii) If $p \in \mathsf{large}$, then $\overline{p} \in \mathsf{small}$.

(ix) If $p \in \mathsf{small}$ and $q \leq_c p$, then $q \in \mathsf{small}$.

(x) If $p \in \mathsf{large}$ and $p \leq_c q$, then $q \in \mathsf{large}$.

(xi) If $p \in \mathsf{half}$ and $q \leq_c p$, then either $q \in \mathsf{small}$ or $q \in \mathsf{half}$.

(xii) If $p \in \mathsf{half}$ and $p \leq_c q$, then either $q \in \mathsf{large}$ or $q \in \mathsf{half}$.

Proof By induction on the number of raw variables in the language. For $n = 0$, the result is trivial.

Assume our result for n, and let the raw variables be $p_0, \ldots, p_{n+1}$. Let Π be a partition for $p_0, \ldots, p_n$. We need to see where p_{n+1} and $\overline{p}_{n+1}$ belong. If any of items (1)–(5) apply, then we know where to put p and $\overline{p}$ using (1)–(5) and (6)–(8). (Since Γ is consistent, at most one of (1)–(5) applies.) In these cases, we must be sure that (9)–(12) continue to hold.

If none of (1)–(5) apply, then we check if any of (9)–(12) apply. More specifically, we take $q = p_{n+1}$ and see if there is some p which works in any of (9)–(12). We first need to check that this step does not lead to contradictory results. And then we would need to know that after we assign p_{n+1} and $\overline{p}_{n+1}$ to the three classes, that (9)–(12) continue to hold. ⊣

We also need two generalizations of Lemma 7.4.1.

We say that the small class is *smaller than* the half class, and both of these are *smaller than* the large class.

Lemma 7.4.2 *Suppose that $\Gamma \not\vdash \exists^{\geq}(p, q)$. Then there is a partition of the nouns as in Lemma 7.4.1 such that one of the following holds:*

(i) p and q are both in small.

(ii) $\overline{p}$ and $\overline{q}$ are both in small.

(iii) p and q are in different classes, and the class of p is smaller than the class of q.

Proof The hypothesis that $\Gamma \not\vdash \exists^{\geq}(p, q)$ implies that Γ is consistent and non-trivial. The idea is to follow the proof of Lemma 7.4.1 but to start with the raw variables underlying p and q. For example, we might classify p as small because $\Gamma \vdash \exists^{\geq}(\overline{p}, p)$, and then perhaps $\overline{p} \leq q$ so that we classify q as large. These first two steps easily arrange that one of the three possibilities in our lemma holds. That is, we cannot classify p and q as both in half, and we also cannot classify p in a class that is larger than the class of q.

Formally, the proof is by induction on the number of raw variables other than the raw variables underlying p and q. If this number is 0, we have sketched the argument. Assume our result for some number n, and suppose that we have $n + 1$ raw variables to classify, and let Π be a partition of the first n (and also of p and q). $\dashv$

7.4.2 Building a model of a consistent set Γ of $\mathcal{S}^\dagger$(card)

Lemma 7.4.3 *Every consistent set Γ has a model.*

We prove this lemma in stages in this section.

The preliminary model $\mathcal{M}_0$ The model $\mathcal{M}_0$ is defined by taking the existential sentences as points in the usual way. Under the standing assumption that Γ is consistent, this model satisfies Γ_{all} and Γ_{some}. As we continue to build other models to satisfy the cardinality sentences in Γ, we need to be sure that the models again satisfy Γ_{all} and Γ_{some}.

$\mathcal{M}_1$: cardinality comparison for small and half At this point, we uncouple our model.

We next use previous work to expand the model to take care of the following types of sentences:

$\exists^{\geq}(p, q)$ with $p \in$ small

$\exists^{>}(p, q)$ with $p \in$ small

$\exists^{\geq}(p, q)$ with $p \in$ half and $q \in$ small

$\exists^{>}(p,q)$ with $p \in \mathsf{half}$ and $q \in \mathsf{small}$

We can do this in such a way that the universal and existential sentences in Γ continue to hold. We treat all of the polarized variables independently, and we need to know that "at the top" (i.e., in half) the order $\leq$ is trivial: see two of the last observations before this paragraph.

We then extend this to take into account comparison between small and half variables. We always insure that the $\forall$ sentences in Γ are respected.

Call this model $\mathcal{M}_1$.

In $\mathcal{M}_1$, if x and y are polarized variables in half, then their interpretations $[\![x]\!]$ and $[\![y]\!]$ might or might not have the same size. Even if they did have the same size, there is no reason to think that this size will be half the size of the universe M_1.

Proposition 7.4.4 *Either Γ is trivial, or else whenever p and q belong to* large*, then it is consistent with Γ that $\exists(p,q)$.*

Proof This is because

$$\dfrac{\forall(p,\overline{q}) \qquad \dfrac{\exists^{\geq}(p,\overline{p}) \quad \exists^{\geq}(q,\overline{q})}{\exists^{\geq}(p,\overline{q})}}{\forall(\overline{q},p)}$$

There are at least as many p as non-p, There are at least as many q as non-q, No p are q $\vdash$ There a

That is, if it were inconsistent with Γ that $\exists(p,q)$, then both p and q would belong to half.

This uses (SUBSET CARD), (CARD TRANS), and (ANTITONE). $\dashv$

$\mathcal{M}_2$**: Adding a point, if necessary** If $|M_1|$ is odd, then we need to add a point. (If $|M_1|$ is even, skip this step.) Let $*$ be a fresh point. We want to insure that if p and $\overline{p}$ belong to half, then the number D_p defined below is even.

$$D_p \quad = \quad |[\![p]\!] - [\![\overline{p}]\!]|.$$

If D_p is odd, then add $*$ to $[\![p]\!]$. (Actually, it would work to add $*$ to either $[\![p]\!]$ or $[\![\overline{p}]\!]$.) We need to insure that Γ_{all} is respected. The first thing is that by the assumption that Γ is non-trivial, $\Gamma \vdash \exists(p,p)$ for all $p \in \mathsf{half}$. Second, we need to say what happens when both p and $\overline{p}$ belong to half and also q and $\overline{q}$. The point is that adding the same point $*$ to both $[\![p]\!]$ and $[\![q]\!]$ insures that $\exists(p,q)$ holds in the model; however, it might be that $\Gamma \vdash \forall(p,\overline{q})$. So for this, we list the raw variables in half:

$$p_1, \ldots, p_k$$

and then we proceed down this list. For $1 \leq i \leq k$, we must either add $*$ to $[\![p_i]\!]$ or $[\![\overline{p}_i]\!]$. And if $i < j$ and $\Gamma \vdash \forall(p_i, p_j)$, then we should not add $*$ to both $[\![p_i]\!]$ and $[\![\overline{p}_j]\!]$. Finally, if $\Gamma \vdash \forall(p_i, \overline{p}_j)$, then we should not add $*$ to both $[\![p_i]\!]$ and $[\![p_j]\!]$. Suppose we have done this up to i, and we consider p_{i+1}. If there is some $j < i+1$ such that $\Gamma \vdash \forall(p_j, p_{i+1})$, then add $*$ to $[\![p_{i+1}]\!]$ iff it was added to $[\![p_j]\!]$. If there is some $j < i+1$ such that $\Gamma \vdash \forall(\overline{p}_j, p_{i+1})$, then add $*$ to $[\![p_{i+1}]\!]$ iff it was added to $[\![\overline{p}_j]\!]$. In other cases, $*$ may be added to either $[\![p_{i+1}]\!]$ or to $[\![\overline{p}_{i+1}]\!]$.

We also would like to know that there cannot be contradictory requirements to fulfill. There are a few cases to consider here, and they are all similar. For example, suppose that we had $j, h < i+1$ such that $p_j \leq p_{i+1}$, $* \in [\![p_j]\!]$, $\overline{p}_h \leq p_{i+1}$, and $* \in [\![p_h]\!]$. If this were possible, we would be in the problematic position of needing to add $*$ to both $[\![p_{i+1}]\!]$ and $[\![\overline{p}_{i+1}]\!]$. However, the assumptions imply that $p_j \leq \overline{p}_h$ (see Example 7.1). Without loss of generality, $j < h$. So our construction has already arranged that $*$ was added to $[\![\overline{p}_h]\!]$. This is a contradiction.

We are still showing that $\mathcal{M}_2$ satisfies all of the *All* sentences in Γ. There are a few more points to check. Notice first that if we are making an addition at this point, then the universe of $\mathcal{M}_1$ must be of odd cardinality, hence non-empty. Thus it must be the case that $\Gamma \vdash \exists(q, q)$ for some q. The (Int) rule tells us that if $p \in \mathsf{half}$, then $\Gamma \vdash \exists(p, p)$. So adding $*$ cannot lead to a contradiction of the form $\forall(p, \overline{p})$ with $p \in \mathsf{half}$. And if p and q are different elements of half, and if Γ contains $\forall(p, \overline{q})$, then $*$ will not be put in both $[\![p]\!]$ and $[\![q]\!]$, by construction. Finally, if p and q are both in large, we might well add $*$ to both $[\![p]\!]$ and $[\![q]\!]$. We claim that in this case, $\Gamma \not\vdash \forall(p, \overline{q})$. For if we did have $p \leq \overline{q}$, then by by Proposition 7.4.4, $\overline{q}$ would belong to large. And this contradicts $q \in \mathsf{large}$.

$\mathcal{M}_3$: Making sure that the half polarized variables are interpreted by half of the universe At this point, we know that the differences D_p are even numbers whenever p is a half-occurring raw variable. Let

$$N \quad = \quad max_p \; D_p$$

By what we just did, N is an even number. We next add N fresh points to the universe, call them $a_1, \ldots, a_N$. When we add points to the universe, we must say for each p whether they go in $[\![p]\!]$ or $[\![\overline{p}]\!]$. If $p \in \mathsf{small}$, we add all the fresh points to $[\![\overline{p}]\!]$. If $p \in \mathsf{large}$, we add all the fresh points to $[\![p]\!]$. If $p \in \mathsf{half}$, we divvy the fresh points up. Suppose that $[\![p]\!] - [\![\overline{p}]\!] \geq 0$, call it m. Then we and $a_1, \ldots, a_{\frac{N+m}{2}}$ to $[\![\overline{p}]\!]$ and $a_{1+\frac{N+m}{2}}, \ldots, a_N$ to $[\![p]\!]$. If $[\![p]\!] - [\![\overline{p}]\!] < 0$, the division is similar. This has arranged that $|[\![p]\!]| = |[\![\overline{p}]\!]|$ for all $p \in \mathsf{half}$. To make sure that these additions have not messed anything up, we need to check that if $\Gamma \vdash \forall(p, q)$, then no a_i belongs to $[\![p]\!] \cap [\![\overline{q}]\!]$. For suppose that this happenend. We have four cases:

(i) $p \in \mathsf{half}$, $\overline{q} \in \mathsf{half}$.

(ii) $p \in \mathsf{half}$, $\overline{q} \in \mathsf{large}$.

(iii) $p \in \mathsf{large}$, $\overline{q} \in \mathsf{half}$.

(iv) $p \in \mathsf{large}$, $\overline{q} \in \mathsf{large}$.

We cannot have $\overline{q}$ in large: for if we did, then $q \in \mathsf{small}$, and so $p \in \mathsf{small}$ as well. This leaves cases 1 and 3. In case 3, we have $q \in \mathsf{half}$, and so this contradicts $p \in \mathsf{large}$ and $\forall(p,q)$. In case 1, we have $p \leq q$, and the construction has arranged that the fresh points in $[\![p]\!]$ are the same as the fresh points in $[\![q]\!]$. So we are again done in this case.

$\mathcal{M}_4$: Taking care of the large variables Finally, we expand $\mathcal{M}_3$ to be sure that all of the cardinality statements for the large variables are true in the model. The only sentences that we need to consider are $\exists^{\geq}(p,q)$ and $\exists^{>}(p,q)$, with $p \in \mathsf{large}$ and $q \in \mathsf{small}$ or $q \in \mathsf{half}$.

We find some appropriate number K and add K fresh points to $[\![p]\!]$ for $p \in \mathsf{large}$. We must be sure that K is even, and then for each $p \in \mathsf{half}$, we add half of the fresh points to $[\![p]\!]$ and half to $[\![\overline{p}]\!]$.

7.4.3 The completeness theorem

Recall that to prove completeness, we are going to show that if $\Gamma \not\vdash \varphi$ then $\Gamma \not\models \varphi$. In logics with (RAA), we would usually consider $\Gamma \cup \{\overline{\varphi}\}$ and then show that this has a model. But in logics like $\mathcal{S}^\dagger$(card), this is not available to us. Instead, we need a generalization of Theorem **??**.

Theorem 7.4.5 *$\Gamma \not\vdash \varphi$ then $\Gamma \not\models \varphi$.*

Proof What we are going to do is to repeat the model constructions in Theorem **??**. We need to argue case-by-case depending on φ.

Case 1: φ is $\forall(p,q)$. One starts $\mathcal{M}_0$ with an extra point declared to be in $[\![p]\!] \cap [\![\overline{q}]\!]$. The rest of the construction is the same.

In case p and q are both small, this follows from what we did on the language of $\exists^{\geq}$. It is easy to check that this same assertion holds for all of the other cases, except when p and q are both in half. Here we need to be sure that adding points to $[\![\overline{p}]\!] \cap [\![q]\!]$ will not cause problems. For if $\Gamma \vdash \forall(\overline{p},\overline{q})$, then also $\Gamma \vdash \forall(q,p)$. But if p and q are in half, then we also have $\Gamma \vdash \forall(p,q)$. And this is a contradiction.

Case 2: φ is $\exists(p,q)$. In this case, we need to be sure that the models $\mathcal{M}_0, \ldots, \mathcal{M}_4$ can be constructed subject to the additional requirement that $[\![p]\!] \cap [\![q]\!] = \emptyset$.

For $\mathcal{M}_0$, this is easy.

For $\mathcal{M}_1$ and $\mathcal{M}_2$, and $p, q \in \mathsf{half}$, it is important that we can always insist that $[\![p]\!] = [\![\overline{q}]\!]$ (and also that $[\![q]\!] = [\![\overline{p}]\!]$). The only way this could cause a problem is that $\Gamma \vdash \exists(p,q)$ (contrary to this case) or $\Gamma \vdash \exists(\overline{p},\overline{q})$. But in this last case, we would have $\Gamma \vdash \exists(p,q)$ using the (Maj) rule.

For $\mathcal{M}_4$, this is due to the fact that if p and q both belong to large, then Γ cannot contain $\forall(p,\overline{q})$.

need

Case 3: φ is $\exists^{\geq}(p,q)$.

Case 4: φ is $\exists^>(p,q)$. We have several subcases.

(i) $\Gamma \nvdash \exists^\geq(p,\overline{p})$, $\Gamma \nvdash \exists^\geq(\overline{p},p)$, $\Gamma \nvdash \exists^\geq(q,\overline{q})$, $\Gamma \nvdash \exists^\geq(\overline{q},q)$.

We can put $p \in \mathsf{small}$ and $q \in \mathsf{large}$.

(ii) $\Gamma \nvdash \exists^\geq(p,\overline{p})$, $\Gamma \nvdash \exists^\geq(\overline{p},p)$, $\Gamma \nvdash \exists^\geq(q,\overline{q})$, $\Gamma \vdash \exists^\geq(\overline{q},q)$.

We can put $p \in \mathsf{small}$ and also put $q \in \mathsf{small}$, and also arrange $\exists^\geq(q,p)$.

(iii) $\Gamma \nvdash \exists^\geq(p,\overline{p})$, $\Gamma \nvdash \exists^\geq(\overline{p},p)$, $\Gamma \vdash \exists^\geq(q,\overline{q})$, $\Gamma \nvdash \exists^\geq(\overline{q},q)$.

We can put $p \in \mathsf{small}$ and we automatically have $q \in \mathsf{large}$.

(iv) $\Gamma \nvdash \exists^\geq(p,\overline{p})$, $\Gamma \nvdash \exists^\geq(\overline{p},p)$, $\Gamma \vdash \exists^\geq(q,\overline{q})$, $\Gamma \vdash \exists^\geq(\overline{q},q)$.

We can put $p \in \mathsf{small}$ and we automatically have $q \in \mathsf{half}$.

(v) $\Gamma \nvdash \exists^\geq(p,\overline{p})$, $\Gamma \vdash \exists^\geq(\overline{p},p)$, $\Gamma \nvdash \exists^\geq(q,\overline{q})$, $\Gamma \nvdash \exists^\geq(\overline{q},q)$.

We automatically have $p \in \mathsf{small}$, and we can put $q \in \mathsf{large}$.

(vi) $\Gamma \nvdash \exists^\geq(p,\overline{p})$, $\Gamma \vdash \exists^\geq(\overline{p},p)$, $\Gamma \nvdash \exists^\geq(q,\overline{q})$, $\Gamma \vdash \exists^\geq(\overline{q},q)$.

We automatically have $p, q \in \mathsf{small}$, and we can arrange that $\exists^\geq(q,p)$.

(vii) $\Gamma \nvdash \exists^\geq(p,\overline{p})$, $\Gamma \vdash \exists^\geq(\overline{p},p)$, $\Gamma \vdash \exists^\geq(q,\overline{q})$, $\Gamma \nvdash \exists^\geq(\overline{q},q)$.

In this case, $\Gamma \vdash \exists^\geq(q,p)$, and so our model $\mathcal{M}_4 \models \neg\exists^>(p,q)$.

(viii) $\Gamma \nvdash \exists^\geq(p,\overline{p})$, $\Gamma \vdash \exists^\geq(\overline{p},p)$, $\Gamma \vdash \exists^\geq(q,\overline{q})$, $\Gamma \vdash \exists^\geq(\overline{q},q)$.

This is the same as case 7.

(ix) $\Gamma \vdash \exists^{\geq}(p,\overline{p})$, $\Gamma \nvdash \exists^{\geq}(\overline{p},p)$, $\Gamma \nvdash \exists^{\geq}(q,\overline{q})$, $\Gamma \nvdash \exists^{\geq}(\overline{q},q)$.

We automatically have $p \in \mathsf{large}$, and we can put $q \in \mathsf{large}$ and also arrange that $p \equiv_c q$.

(x) $\Gamma \vdash \exists^{\geq}(p,\overline{p})$, $\Gamma \nvdash \exists^{\geq}(\overline{p},p)$, $\Gamma \nvdash \exists^{\geq}(q,\overline{q})$, $\Gamma \vdash \exists^{\geq}(\overline{q},q)$.

(xi) $\Gamma \vdash \exists^{\geq}(p,\overline{p})$, $\Gamma \nvdash \exists^{\geq}(\overline{p},p)$, $\Gamma \vdash \exists^{\geq}(q,\overline{q})$, $\Gamma \nvdash \exists^{\geq}(\overline{q},q)$.

(xii) $\Gamma \vdash \exists^{\geq}(p,\overline{p})$, $\Gamma \nvdash \exists^{\geq}(\overline{p},p)$, $\Gamma \vdash \exists^{\geq}(q,\overline{q})$, $\Gamma \vdash \exists^{\geq}(\overline{q},q)$.

(xiii) $\Gamma \vdash \exists^{\geq}(p,\overline{p})$, $\Gamma \vdash \exists^{\geq}(\overline{p},p)$, $\Gamma \nvdash \exists^{\geq}(q,\overline{q})$, $\Gamma \nvdash \exists^{\geq}(\overline{q},q)$.

We automatically have $p \in \mathsf{half}$, and we can arrange that $q \in \mathsf{large}$.

(xiv) $\Gamma \vdash \exists^{\geq}(p,\overline{p})$, $\Gamma \vdash \exists^{\geq}(\overline{p},p)$, $\Gamma \nvdash \exists^{\geq}(q,\overline{q})$, $\Gamma \vdash \exists^{\geq}(\overline{q},q)$.

We automatically have $p \in \mathsf{half}$, and **we should be able to** put $q \in \mathsf{half}$ as well.

(xv) $\Gamma \vdash \exists^{\geq}(p,\overline{p})$, $\Gamma \vdash \exists^{\geq}(\overline{p},p)$, $\Gamma \vdash \exists^{\geq}(q,\overline{q})$, $\Gamma \nvdash \exists^{\geq}(\overline{q},q)$.

We automatically have $p \in \mathsf{half}$ and $q \in \mathsf{large}$.

(xvi) $\Gamma \vdash \exists^{\geq}(p,\overline{p})$, $\Gamma \vdash \exists^{\geq}(\overline{p},p)$, $\Gamma \vdash \exists^{\geq}(q,\overline{q})$, $\Gamma \vdash \exists^{\geq}(\overline{q},q)$.

In $\mathcal{M}_4$, p and q have the same size.

This completes the proof of Theorem 7.4.5. $\dashv$

7.5 Adding $\exists^{\geq}$ to the Boolean syllogistic fragment

We now put aside *Most* and return to the study of $\exists^{\geq}$ from earlier. We close this chapter with the addition of $\exists^{\geq}$ to the fragment of Section 7.

Our logical system extends the axioms of Figure **??** by those in Figure **??**. Note that the last new axiom expresses *cardinal comparison*. Axiom 4 in Figure 7.3 is just a transcription of the rule for *No* that we saw in Section 42. We do not need to also add the axiom

$$(\textit{Some } y \textit{ are } y) \land \exists^{\geq}(x,y) \rightarrow \textit{Some } x \textit{ are } x$$

because it is derivable. Here is a sketch, in English. Assume that there are some ys, and there are at least as many xs as ys, but (towards a contradiction) that there are no xs. Then all x's are ys. From our logic, all ys are xs as well. And since there are y's, there are also x's: a contradiction.

Notice also that in the current fragment we can express *There are more x than y*. It would be possible to add this directly to our previous systems.

Theorem 7.5.1 *The logic of Figures* **??** *and 7.3 is complete for assertions $\Delta \models \varphi$ in the language of boolean combinations of sentences in $\mathcal{L}(\mathsf{all}, \mathsf{some}, \mathsf{no}, \exists^{\geq})$.*

(i) *All* x *are* $y \to \exists^{\geq}(y,x)$

(ii) $\exists^{\geq}(x,y) \land \exists^{\geq}(y,z) \to \exists^{\geq}(x,z)$

(iii) *All* y *are* $x \land \exists^{\geq}(y,x) \to$ *All* x *are* y

(iv) *No* x *are* $x \to \exists^{\geq}(y,x)$

(v) $\exists^{\geq}(x,y) \lor \exists^{\geq}(y,x)$

Figure 7.3: Additions to the system in Figure **??** for $\exists^{\geq}$ sentences.

Proof We need only build a model for a maximal consistent set Δ in the language of this section. We take the *basic* sentences to be those of the form *All* x *are* y, *Some* x *and* y, J *is* M, J *is an* x, $\exists^{\geq}(x,y)$, or their negations. Let

$$\Gamma = \{S : \Delta \models S \text{ and } \varphi \text{ is basic}\}.$$

As in Section 4.4, we need only build a model $\mathcal{M} \models \Gamma$ (see Lemma **??**). We construct $\mathcal{M}$ such that for all A and B, rework this

(α) $[\![A]\!] \subseteq [\![B]\!]$ iff $A \leq B$.

(β) $A \leq_c B$ iff $|[\![A]\!]| \leq |[\![B]\!]|$.

(γ) For $A \leq_c B$, $[\![A]\!] \cap [\![B]\!] \neq \emptyset$ iff $A \uparrow B$.

Let $\mathcal{V}$ be the set of variables in Γ. Let $\leq_c$ and $\equiv_c$ be as in Section **??**. Proposition 7.1.1 again holds, and now the quotient $\mathcal{V}/\equiv_c$ is a linear order due to the last axiom in Figure 7.3. We write it as

$$[u_0] \quad <_c \quad [u_2] \quad <_c \quad \cdots \quad <_c \quad [u_k]$$

We define by recursion on $i \leq k$ the interpretation $[\![v]\!]$ of all $v \in [u_i]$. The case of $i = 0$ is special. If $\Gamma \models$ *No* u_0 *is a* u_0, then the same holds for all $w \equiv_c u_0$. In this case, we set $[\![w]\!] = \emptyset$ for all these w. Note that by our fourth axiom in Figure 7.3, all of the other variables w are such that $\Gamma \vdash \exists w$. In any case, we must interpret the variables in $[u_0]$ even when $\Gamma \vdash (\exists u_0)$. In this case, we may take each $[\![w]\!]$ to be a singleton, with the added condition that $v \equiv w$ iff $[\![v]\!] = [\![w]\!]$.

Suppose we have $[\![w]\!]$ for all $j \leq i$ and all $w \equiv_c u_j$. Let

$$\mathcal{X}_{i+1} = \bigcup_{j \leq i, w \equiv_c u_j} [\![w]\!],$$

and note that this is the set of all points used in the semantics of any variable so far. Let $m = |\Gamma_{some}|$, and let

$$n = 1 + m + |\mathcal{X}_{i+1}| \tag{7.1}$$

For all $v \equiv_c u_{i+1}$, we shall arrange that $[\![v]\!]$ be a set of size n.

Now $[u_{i+1}]$ splits into equivalence classes of the finer relation $\equiv$. For a moment, consider one of those finer classes, say $[A]_\equiv$. We must interpret each variable in this class by the same set. For this A, let

$$\mathcal{Y}_A \quad = \quad \bigcup\{[\![B]\!] : (\exists j \leq i)\ v_j \equiv_c B \leq A\}.$$

Note that $\mathcal{Y}_A \subseteq \mathcal{X}_{i+1}$ so that $|\mathcal{Y}_A| \leq |\mathcal{X}_{i+1}|$ for all $A \equiv_c u_{i+1}$. We shall set $[\![A]\!]$ to be $\mathcal{Y}_A$ plus other points. Let ${}_A$ be the set of pairs $\{A, B\}$ with $B \equiv_c u_{i+1}$ and $A \uparrow_\Gamma B$. (This is the same as saying that *Some A are B* in Γ_{some}.) Notice that if both A and B are $\equiv_c u_{i+1}$ and $A \uparrow_\Gamma B$, then $\{A, B\} \in \mathcal{Z}_A \cap {}_B$.) We shall set $[\![A]\!]$ to be $\mathcal{Y}_A \cup {}_A$ plus one last group of points. If $C <_c u_{i+1}$ and $A \uparrow_\Gamma C$, then we must pick some element of $[\![C]\!]$ and put it into $[\![A]\!]$. Note that the number of points selected like this plus $|\mathcal{Z}_A|$ is still $\leq |\Gamma_{some}|$. So the number of points so far in $[\![A]\!]$ is $\leq |\Gamma_{some}| + m$. We finally add fresh elements to $[\![A]\!]$ so that the total is n.

We do all of this for all of the other $\equiv$-classes which partition the $\equiv_c$-class of u_{i+1}. We must insure that for $A \not\equiv A'$, the fresh elements added into $[\![A']\!]$ are disjoint from the fresh elements added into $[\![A]\!]$. This is needed to arrange that neither $[\![A]\!]$ nor $[\![A']\!]$ will be a subset of the other.

This completes the definition of the model. We say a few words about why requirements (α)–(γ) are met. First, and easy induction on i shows that if $j < i$, then $|[\![u_j]\!]| < |[\![u_i]\!]|$. The point is that $|[\![u_j]\!]| \leq |\mathcal{X}_i| < |[\![u_i]\!]|$. The argument for (β) is the same as in the proof of Theorem **??**. For that matter, the proof of (α) is also essentially the same. The point is that when $A \equiv_c B$ and $A \not\equiv B$, then $[\![A]\!]$ and $[\![B]\!]$ each contain a point not in the other.

For (γ), suppose that $A \leq_c B$. Let $i \leq j$ be such that $A \equiv_c u_i$ and $B \equiv u_j$. The construction arranged that $[\![A]\!]$ and $[\![B]\!]$ be disjoint except for the case that $A \uparrow B$.

So this verifies that (α)–(γ) hold. We would like to conclude that $\mathcal{M} \models \Gamma$, but there is one last point: (γ) appears to be a touch too weak. We need to know that $[\![A]\!] \cap [\![B]\!] \neq \emptyset$ iff $A \uparrow B$ (without assuming $A \leq_c B$). But either $A \leq_c B$ or $B \leq_c A$ by our last axiom. So we see that indeed $[\![A]\!] \cap [\![B]\!] \neq \emptyset$ iff $A \uparrow B$. $\dashv$

The next step in this direction would be to consider *At least as many x as y are z*.

7.6 Most

The semantics of *Most* is that *Most x are y* are that this is true iff $|[\![x]\!] \cap [\![y]\!]| > \frac{1}{2}|[\![x]\!]|$. So if $[\![x]\!]$ is empty, then *Most x are y* is false.

As an example of what is going on, consider the following. Assume that *All x are z*, *All y are z*, *Most z are y*, and *Most y are x*. Does it follow that *Most x are y*? As it happens, the conclusion does not follow. One can take $x = \{a, b, c, d, e, f, g\}$, $y = \{e, f, g, h, i\}$, and $z = \{a, b, c, d, e, f, g, h, i\}$. Then $|x| = 7$, $|y| = 5$, $|z| = 9$, $|y \cap z| = 5 > 9/5$, $|x \cap y| = 3 > 5/2$, but $|x \cap y| = 3 < 7/2$. (Another countermodel: let $x = \{1, 2, 4, 5\}$, $y = \{1, 2, 3\}$, and $z = \{1, 2, 3, 4, 5\}$. Then $|y \cap z| = 3 > 5/2$, $|y \cap x| = 2 > 3/2$, but $|x \cap y| = 2 \not> 4/2$.)

$$\frac{\textit{Most } x \textit{ are } y}{\textit{Some } x \textit{ are } y} \quad \frac{\textit{Some } x \textit{ are } x}{\textit{Most } x \textit{ are } x} \quad \frac{\textit{Most } x \textit{ are } y \quad \textit{Most } x \textit{ are } z}{\textit{Some } y \textit{ are } z}$$

Figure 7.4: Rules of *Most* to be used in conjunction with *Some*.

On the other hand, the following is a sound rule:

$$\frac{\textit{All } u \textit{ are } x \quad \textit{Most } x \textit{ are } v \quad \textit{All } v \textit{ are } y \quad \textit{Most } y \textit{ are } u}{\textit{Some } u \textit{ are } v}$$

Here is the reason for this. Assume our hypotheses and also that towards a contradiction that u and v were disjoint. We obviously have $|v| \geq |x \cap v|$, and the second hypothesis, together with the disjointness assumption, tells us that $|x \cap v| > |x \cap u|$. By the first hypothesis, we have $|x \cap u| = |u|$. So at this point we have $|v| > |u|$. But the last two hypotheses similarly give us the opposite inequality $|u| > |v|$. This is a contradiction.

At the time of this writing, I do not have a completeness result for $\mathcal{L}(\mathsf{all}, \mathsf{some}, \mathsf{most})$. The best that is known is for $\mathcal{L}(\mathsf{some}, \mathsf{most})$. The rules are are shown in Figure 7.4. We study these on top of the rules for *some* which we have seen:

$$\frac{\textit{Some } y \textit{ are } x}{\textit{Some } x \textit{ are } y} \qquad \frac{\textit{Some } x \textit{ are } y}{\textit{Some } x \textit{ are } x} \tag{7.2}$$

Proposition 7.6.1 *The following two axioms are complete for* Most*.*

$$\frac{\text{Most } x \text{ are } y}{\text{Most } x \text{ are } x} \qquad \frac{\text{Most } x \text{ are } y}{\text{Most } y \text{ are } y}$$

Moreover, if $\Gamma \subseteq \mathcal{L}(\mathsf{most})$*,* $x \neq y$*, and* $\Gamma \not\models$ Most x are y*, then there is a model* $\mathcal{M}$ *of* Γ *which falsifies* Most x are y *in which all sets of the form* $[\![u]\!] \cap [\![v]\!]$ *are nonempty, and* $|M| \leq 5$*.*

Proof Suppose that $\Gamma \not\vdash$ *Most x are y*. We construct a model $\mathcal{M}$ which satisfies all sentences in Γ, but which falsifies *Most x are x*. There are two cases. If $x = y$, then x does not occur in any sentence in Γ. We let $\mathcal{M} = \{*\}$, $[\![x]\!] = \emptyset$, and $[\![y]\!] = \{*\}$ for $y \neq x$.

The other case is when $x \neq y$. Let $\mathcal{M} = \{1, 2, 3, 4, 5\}$, $[\![x]\!] = \{1, 2, 4, 5\}$, $[\![y]\!] = \{1, 2, 3\}$, and for $z \neq x, y$, $[\![z]\!] = \{1, 2, 3, 4, 5\}$. Then the only statement in *Most* which fails in the model $\mathcal{M}$ is *Most x are y*. But this sentence does not belong to Γ. Thus $\mathcal{M} \models \Gamma$. ⊣

Theorem 7.6.2 *The rules in Figure 7.4 together with the first two rules (7.2) are complete for* $\mathcal{L}(\mathsf{some}, \mathsf{most})$*. Moreover, if* $\Gamma \not\models \varphi$*, then there is a model* $\mathcal{M} \models \Gamma$ *with* $\mathcal{M} \not\models \varphi$*, and* $|M| \leq 6$*.*

Proof Suppose $\Gamma \not\vdash \varphi$, where φ is *Some x are y*. If $x = y$, then Γ contains no sentence involving x. So we may satisfy Γ and falsify φ in a one-point model, by setting $[\![x]\!] = \emptyset$ and $[\![z]\!] = \{*\}$ for $z \neq x$.

We next consider the case when $x \neq y$. Then Γ does not contain φ, *Some y are x*, *Most x are y*, or *Most y are x*. And for all z, Γ does not contain both *Most z are x* and *Most z are y*. Let $M = \{1,2,3,4,5,6\}$, and consider the subsets $a = \{1,2,3\}$, $b = \{1,2,3,4,5\}$, $c = \{2,3,4,5,6\}$, and $d = \{4,5,6\}$. Let $[\![x]\!] = a$ and $[\![y]\!] = d$, so that $\mathcal{M} \not\models \varphi$. For z different from x and y, if Γ does not contain *Most z are x*, let $[\![z]\!] = c$. Otherwise, Γ does not contain *Most z are y*, and so we let $[\![z]\!] = b$. For all these z, $\mathcal{M}$ satisfies whichever of the sentences *Most z are x* and *Most z are y* (if either) which belong to Γ. $\mathcal{M}$ also satisfies all sentences *Most x are z* and *Most y are z*, whether or not these belong to Γ. It also satisfies *Most u are u* for all u. Also, for z, z' each different from both x and y, $\mathcal{M} \models$ *Most z are z′*. Finally, $\mathcal{M}$ satisfies all sentences *Some u are v* except for $u = x$ and $y = v$ (or vice-versa). But those two sentences do not belong to Γ. The upshot is that $\mathcal{M} \models \Gamma$ but $\mathcal{M} \not\models \varphi$.

Up until now in this proof, we have considered the case when φ is *Some x are y*. We turn our attention to the case when φ is *Most x are y*. Suppose $\Gamma \nvdash \varphi$. If $x = y$, then the second rule of Figure 7.4 shows that $\Gamma \nvdash$ *Some x are x*. So we take $M = \{*\}$ and take $[\![x]\!] = \emptyset$ and for $y \neq x$, $[\![y]\!] = M$. It is easy to check that $\mathcal{M} \models \Gamma$.

Finally, if $x \neq y$, we clearly have $\Gamma_{most} \nvdash \varphi$. Proposition 7.6.1 shows that there is a model $\mathcal{M} \models \Gamma_{most}$ which falsifies φ in which all sets of the form $[\![u]\!] \cap [\![v]\!]$ are nonempty. So all *Some* sentences hold in $\mathcal{M}$. Hence $\mathcal{M} \models \Gamma$. ⊣

7.7 The numerical syllogistic

I hope to add Ian Pratt-Hartmann's results in [**?**] on the complexity of reasoning with the numerical syllogistic, and also his result in [**?**] that there is no finite syllogistic system for it.

Sources for this chapter The material in this chapter comes from Moss [**?**].

7.8 Exercises

Exercise 41. Given an example which shows a semantic failure of the Compactness Theorem $\mathcal{S}$(card). That is, find an infinite set Γ with the property that every finite subset of Γ has a model, but Γ itself has no model.

Exercise 42. Consider the following two rules:

$$\frac{\textit{Some } y \textit{ are } y \quad \exists^{\geq}(x,y)}{\textit{Some } x \textit{ are } x} \qquad \frac{\textit{No } y \textit{ are } y}{\exists^{\geq}(x,y)}$$

(i) Add the rule on the left to our existing system for $\mathcal{S}(\mathsf{all}, \mathsf{some})$ and to the rules in Figure **??**. Prove that the resulting system is complete for $\mathcal{L}(\mathsf{all}, \mathsf{some}, \exists^{\geq})$.

(ii) Similarly, add the rule on the right to get a complete logic for the resulting language.

(iii) Finally, add both rules to again get a complete logic for the resulting language.

8 Small Additions

8.1 Adding names

8.2 Adding Boolean connectives to sentences

The purpose of this section is to add to our logical systems for $\mathcal{S}^\dagger$ and $\mathcal{RCA}$ the standard Boolean sentential connectives: not ($\neg$), and ($\wedge$), or ($\vee$), if ($\rightarrow$), if and only if ($\leftrightarrow$). So we are adding to the syntax; the sentences of $\mathcal{S}^\dagger$, and $\mathcal{RCA}$ are now regarded as *atomic* sentences, and we add the Boolean connectives on top of our previous work. The semantics is the obvious one.

For the proof theory, we wish to present a proof system for the enlarged languages which extends our previous systems.

We continue to present our system as a natural deduction system. We could also present it as a Fitch-style system as in textbook presentations.

9 The Limits of Syllogistic Proof Systems

When we presented a logic for $\mathcal{RCA}$ in Section 6, we made use of *reductio ad absurdum* (RAA) . The reader might well wonder whether we *need* (RAA) or some such device, or whether the kinds of logics which we have previously considered were sufficient. Now is the time to answer this. We need to formulate exactly what we mean by a *purely syllogistic system*, and then prove that there are no purely syllogistic systems for $\mathcal{R}$ which are finite, sound, and complete.

9.1 General definitions on syllogistic proof systems

Let $\mathcal{F}$ be a syllogistic fragment. A *derivation relation* $\mid\!\sim$ in $\mathcal{F}$ is a subset of $\mathbb{P}(\mathcal{F}) \times \mathcal{F}$, where $\mathbb{P}(\mathcal{F})$ is the power set of $\mathcal{F}$. For readability, we write $\Theta \mid\!\sim \theta$ instead of $\langle \Theta, \theta \rangle \in \mid\!\sim$. We say that $\mid\!\sim$ is *sound* if $\Theta \mid\!\sim \theta$ implies $\Theta \models \theta$, and *complete* (*for* $\mathcal{F}$) if $\Theta \models \theta$ implies $\Theta \mid\!\sim \theta$.

Definition Let $\mathcal{F}$ be a syllogistic fragment, what we have been calling a *logical language* in these notes. We employ the following terminology. A *syllogistic rule* (sometimes, simply: *rule*) in $\mathcal{F}$ is a pair Θ/θ, where Θ is a finite set (possibly empty) of $\mathcal{F}$-sentences, and θ an $\mathcal{F}$-sentence. We call Θ the *antecedents* of the rule, and θ its *consequent*. The rule Θ/θ is *sound* if $\Theta \models \theta$.

A *substitution* is a function $g = g_1 \cup g_2$, where $g_1 : \mathbf{P} \to \mathbf{P}$ and $g_2 : \mathbf{R} \to \mathbf{R}$. If θ is an $\mathcal{F}$-formula, denote by $g(\theta)$ the $\mathcal{F}$-formula which results by replacing any atom (unary or binary) in θ by its image under g, and similarly for sets of formulas. An *instance* of a syllogistic rule Θ/θ is the syllogistic rule $g(\Theta)/g(\theta)$, where g is a substitution.

A *syllogistic proof system* is a set of syllogistic rules.

We generally display rules in 'natural-deduction' style. For example,

$$\frac{\forall(q,o) \qquad \exists(p,q)}{\exists(p,o)} \qquad\qquad \frac{\forall(q,\bar{o}) \qquad \exists(p,q)}{\exists(p,\bar{o})} \tag{9.1}$$

where p, q and o are unary atoms, are syllogistic rules in $\mathcal{S}$, corresponding to the traditional syllogisms *Darii* and *Ferio*, respectively.

Syllogistic rules which differ only with respect to re-naming of unary or binary atoms will be informally regarded as identical, because they have the same instances. Thus, the letters p, q and o in (9.1) function, in effect, as *variables* ranging over unary atoms. It is often convenient to display syllogistic rules using variables ranging over other types of expressions, understanding that these are just more compact ways of writing finite collections of syllogistic rules in the official sense. For example, the two rules (9.1) may be more compactly written

$$\frac{\forall(q,l) \qquad \exists(p,q)}{\exists(p,l)} \text{ (D1)}$$

where p and q range over unary atoms, but l ranges over unary *literals*.

Fix a syllogistic fragment $\mathcal{F}$, and let $\mathcal{X}$ be a set of syllogistic rules in $\mathcal{F}$. Define $\vdash_{\mathcal{X}}$ to be the smallest derivation relation in $\mathcal{F}$ satisfying:

(i) if $\theta \in \Theta$, then $\Theta \vdash_{\mathcal{X}} \theta$;

(ii) if $\{\theta_1, \ldots, \theta_n\}/\theta$ is a rule in $\mathcal{X}$, g a substitution, $\Theta = \Theta_1 \cup \cdots \cup \Theta_n$, and $\Theta_i \vdash_{\mathcal{X}} g(\theta_i)$ for all i $(1 \leq i \leq n)$, then $\Theta \vdash_{\mathcal{X}} g(\theta)$.

It is simple to show that the derivation relation $\vdash_{\mathcal{X}}$ is sound if and only if each rule in $\mathcal{X}$ is sound.

Informally, we imagine chaining together instances of the rules in $\mathcal{X}$ to construct *derivations*, in the obvious way; and we refer to the resulting proof system as the *direct syllogistic system defined by* $\mathcal{X}$. We generally display derivations in natural-deduction style, as we have been doing throughout these notes.

9.2 No systems (without RAA) for $\mathcal{R}$

Theorem 9.2.1 ([?]) *There exists no finite set $\mathcal{X}$ of syllogistic rules in $\mathcal{R}$ such that $\vdash_{\mathcal{X}}$ is both sound and complete.*

Proof Let $\mathcal{X}$ be any finite set of syllogistic rules for $\mathcal{R}$, and suppose $\vdash_{\mathcal{X}}$ is sound. We show that it is not complete. Since $\mathcal{X}$ is finite, fix $n \in N$ greater than the number of antecedents in any rule in $\mathcal{X}$.

Let $p_1, \ldots, p_n$ be distinct unary atoms and r a binary atom. Let Γ be the following set of $\mathcal{R}$-formulas:

$$\forall(p_i, \exists(p_{i+1}, r)) \qquad (1 \leq i < n) \tag{9.2}$$
$$\forall(p_1, \forall(p_n, r)) \tag{9.3}$$
$$\forall(p, p) \qquad (p \in \mathbf{P}) \tag{9.4}$$
$$\forall(p_i, \bar{p}_j) \qquad (1 \leq i < j \leq n) \tag{9.5}$$

and let γ be the $\mathcal{R}$-formula $\forall(p_1, \exists(p_n, r))$. Observe that $\Gamma \models \gamma$. To see this, let $\mathcal{M} \models \Gamma$. If $p_1^{\mathcal{M}} = \emptyset$, then trivially $\mathcal{M} \models \gamma$; on the other hand, if $p_1^{\mathcal{M}} \neq \emptyset$, a simple induction using formulas (9.2) shows that $p_i^{\mathcal{M}} \neq \emptyset$ for all i $(1 \leq i \leq n)$, whence $\mathcal{M} \models \gamma$ by (9.3).

For $1 \leq i < n$, let $\Delta_i = \Gamma \setminus \{\forall(p_i, \exists(p_{i+1}, r))\}$.

Claim If $\varphi \in \mathcal{R}$ and $\Delta_i \models \varphi$, then $\varphi \in \Gamma$.

It follows from this claim that $\Gamma \nvdash_{\mathcal{X}} \gamma$. For, since no rule of $\mathcal{X}$ has more than $n-1$ antecedents, any instance of those antecedents contained in Γ must be contained in Δ_i for some i. Let δ be the corresponding instance of the consequent of that rule. Since $\vdash_{\mathcal{X}}$ is sound, $\Delta_i \models \delta$. By Claim 9.2, $\delta \in \Gamma$. By induction on the number of steps in derivations, we see that no derivation from Γ leads to a formula not in Γ. But $\gamma \notin \Gamma$.

Proof [Proof of Claim] Certainly, Δ_i has a model, for instance the model $\mathcal{M}_i$ given by:

$$\begin{tikzcd}
p_1 \arrow[r] \arrow[rrrrrr, bend left] & p_2 \arrow[r] & \cdots \arrow[r] & p_i & p_{i+1} \arrow[r] & \cdots \arrow[r] & p_n
\end{tikzcd} \tag{9.6}$$

Here, $A = \{p_1, \ldots, p_n\}$, $p_j^{\mathcal{M}_i} = \{p_j\}$ for all j $(1 \leq j \leq n)$, and $r^{\mathcal{M}_i}$ is indicated by the arrows. All other atoms (unary or binary) are assumed to have empty extensions. Note that there is no arrow from p_i to p_{i+1}.

We consider the various possibilities for φ in turn and check that either $\varphi \in \Gamma$ or there is a model of Δ_i in which φ is false.

(i) φ is of the form $\forall(p,p)$. Then $\varphi \in \Gamma$ by (9.4).

(ii) φ is not of the form $\forall(p,p)$, and involves at least one unary or binary atom other than $p_1, \ldots, p_n, r$. In this case, it is straightforward to modify $\mathcal{M}_i$ so as to obtain a model $\mathcal{M}'_i$ of Δ_i such that $\mathcal{M}'_i \not\models \varphi$. Henceforth, then, we may assume that φ involves no atoms other than $p_1, \ldots, p_n, r$.

(iii) φ is of the form $\forall(p_j, p_k)$. If $j = k$, then $\varphi \in \Gamma$, by (9.4). If $j \neq k$, then $\mathcal{M}_i \not\models \varphi$ by inspection.

(iv) φ is of the form $\forall(p_j, \bar{p}_k)$. If $j = k$, then $\mathcal{M}_i \not\models \varphi$, since $p_j^{\mathcal{M}_i} \neq \emptyset$. If $j \neq k$, then $\varphi \in \Gamma$, by (9.5) and the identification $\forall(p_j, \bar{p}_k) = \forall(p_k, \bar{p}_j)$.

(v) φ is of the form $\forall(p_j, \forall(p_k, r))$. If $j = 1$ and $k = n$, then $\varphi \in \Gamma$, by (9.3). So we may assume that either $j > 1$ or $k < n$, in which case, $k \neq j+1$ implies $\mathcal{M}_i \not\models \varphi$, by inspection. Hence, we may assume that $\varphi = \forall(p_j, \forall(p_{j+1}, r))$, with $j < n$. Let $\mathfrak{B}_{i,j}$ be the structure obtained from $\mathcal{M}_i$ by adding a second point b to the interpretation of p_{j+1}, and to which p_j is not related by r. In pictures:

$$\begin{tikzcd}
p_1 \arrow[r] \arrow[rrrrrrrrrr, bend left] & p_2 \arrow[r] & \cdots & p_j \arrow[r] & p_{j+1} \arrow[r] & p_{j+2} \arrow[r] & \cdots \arrow[r] & p_i & p_{i+1} \arrow[r] & \cdots & p_n \\
 & & & & b \arrow[ur] & & & & & &
\end{tikzcd}$$

(This picture shows $j+2 < i$. Similar pictures are possible in all other cases.) By inspection, $\mathfrak{B}_{i,j} \models \Delta_i$, but $\mathfrak{B}_{i,j} \not\models \varphi$.

(vi) φ is of the form $\forall(p_j, \exists(p_k, r))$. If $k = j+1$, then $\varphi \in \Gamma$, by (9.2). Moreover, if $k \neq j+1$, then, unless $j = 1$ and $k = n$, $\mathcal{M}_i \not\models \varphi$, by inspection. Hence we may assume $\varphi = \forall(p_1, \exists(p_n, r))$. Let $\mathfrak{C}_i$ be the structure:

$$\begin{tikzcd}
p_1 \arrow[r] & p_2 \arrow[r] & \cdots \arrow[r] & p_i,
\end{tikzcd}$$

with $p_j^{\mathfrak{C}_i} = \emptyset$ for all j $(i < j \leq n)$. Then $\mathfrak{C}_i \models \Delta_i$, but $\mathfrak{C}_i \not\models \varphi$.

(vii) φ is of either of the forms $\forall(p_j, \forall(p_k, \bar{r}))$, $\forall(p_j, \exists(p_k, \bar{r}))$. Define $\mathcal{M}_i''$ to be like $\mathcal{M}_i$ except that $r^{\mathcal{M}_i''}$ additionally contains the pair of points $\langle p_j, p_k \rangle$. By inspection, $\mathcal{M}_i'' \models \Delta_i$, but $\mathcal{M}_i'' \not\models \varphi$.

(viii) φ is of the form $\exists(p, c)$. Let $\mathcal{M}_0$ be a structure over any domain in which every atom has empty extension. Then $\mathcal{M}_0 \models \Delta_i$, but $\mathcal{M}_0 \not\models \varphi$. $\dashv$

This also completes the proof of Theorem 9.2.1. $\dashv$

9.3 No systems (even with RAA) for $\mathcal{RC}$ or $\mathcal{RC}^\dagger$

10 Logic Beyond the Aristotle Border

There are only two languages in this chapter (actually they are families of languages parameterized by sets of basic symbols): the language $\mathcal{RC}^{\dagger}$ which we have seen already, and an extension $\mathcal{L}(adj)$ studied in Section 10.7. We reformulate $\mathcal{RC}^{\dagger}$ a bit, adding constants and also being allowing for recursive constructs. To avoid confusion, we do not speak of $\mathcal{RC}^{\dagger}$ but instead call the language of this chapter $\mathcal{L}$. $\mathcal{L}$ is based on three pairwise disjoint sets called $\mathbf{P}$, $\mathbf{R}$, and $\mathbf{K}$. These are called *unary atoms*, *binary atoms*, and *constant symbols*.

10.1 Fitch-style proof system for $\mathcal{RC}^{\dagger}$

We review the syntax of $\mathcal{L}$ in Figure 10.1. Sentences are built from constant symbols, unary and binary atoms using an involutive symbol for negation, a formation of set terms, and also a form of quantification. The second column indicates the variables that we shall use in order to refer to the objects of the various syntactic categories. Because the syntax is not standard, it will be worthwhile to go through it slowly and to provide glosses in English for expressions of various types.

One might think of the constant symbols as proper names such as John and Mary. The unary atoms may be glossed as one-place predictates such as boys, girls, etc. And the relation symbols correspond to transitive verbs (that is, verbs which take a direct object) such as likes, sees, etc. They also correspond to comparative adjective phrases such as is bigger than. (However, later on in Section 10.7, we introduce a new syntactic primitive for the adjectives.)

Unary atoms *appear to be* one-place relation symbols, especially because we shall form sentences of the form $p(j)$. However, we do not have sentences $p(x)$, since we have no variables at this point in the first place. Similar remarks apply to binary atoms and two-place relation symbols. So we chose to change the terminology from *relation symbols* to *atoms*.

We form unary and binary *literals* using the bar notation. We think of this as expressing classical negation. So we take it to be involutive, so that $\overline{\overline{p}} = p$ and $\overline{\overline{s}} = s$.

The set terms in this language are the only recursive construct. If b is read as boys and s as sees, then one should read $\forall(b, s)$ as sees all boys, and $\exists(b, s)$ as sees some boys. Hence these set terms correspond to simple verb phrases. We also allow negation on the atoms, so we have $\forall(b, \overline{s})$; this can be read as fails to see all boys, or (better) sees no boys or doesn't see any boys. We also have $\exists(b, \overline{s})$, fails to see some boys. But the recursion allows us to embed set terms, and so we have set terms like

$$\exists(\forall(\forall(b, \overline{s}), h), a)$$

Expression	Variables	Syntax
unary atom	p, q	
binary atom	s	
constant	j, k	
unary literal	l	$p \mid \bar{p}$
binary literal	r	$s \mid \bar{s}$
set term	b, c, d	$l \mid \exists(c,r) \mid \forall(c,r)$
sentence	φ, ψ	$\forall(c,d) \mid \exists(c,d) \mid c(j) \mid r(j,k)$

Figure 10.1: Syntax of sentences of $\mathcal{L}$.

which may be taken to symbolize a verb phrase such as admires someone who hates everyone who does not see any boy.

We should note that the relative clauses which can be obtained in this way are all "missing the subject", never "missing the object". The language is too poor to express predicates like $\lambda x.$all boys see x.

The main sentences in the language are of the form $\forall(b,c)$ and $\exists(b,c)$; they can be read as statements of the inclusion of one set term extension in another, and of the non-empty intersection. We also have sentences using the constants, such as $\forall(g,s)(m)$, corresponding to Mary sees all girls. But we are not able to say all girls see Mary; the syntax again is too weak. (However, in our Conclusion we shall see how to extend our system to handle this.) This weakness in expressive power corresponds to a less complex decidability result, as we shall see.

Semantics. A *model* (for this language $\mathcal{L}$) is a pair $\mathcal{M} = \langle M, [\![\ ]\!]\rangle$, where M is a non-empty set, $[\![p]\!] \subseteq M$ for all $p \in \mathbf{P}$, $[\![r]\!] \subseteq M^2$ for all $r \in \mathbf{R}$, and $[\![j]\!] \in M$ for all $j \in \mathbf{K}$.

Given a model $\mathcal{M}$, we extend the interpretation function $[\![\ ]\!]$ to the rest of the language by setting

$$\begin{array}{lcl} [\![\bar{p}]\!] & = & M \setminus [\![p]\!] \\ [\![\bar{r}]\!] & = & M^2 \setminus [\![r]\!] \\ [\![\exists(l,t)]\!] & = & \{x \in M : \text{ for some } y \text{ such that } [\![l]\!](y),\ [\![t]\!](x,y)\} \\ [\![\forall(l,t)]\!] & = & \{x \in M : \text{ for all } y \text{ such that } [\![l]\!](y),\ [\![t]\!](x,y)\} \end{array}$$

We define the truth relation $\models$ between models and sentences by:

$$\begin{array}{lll} \mathcal{M} \models \forall(c,d) & \text{iff} & [\![c]\!] \subseteq [\![d]\!] \\ \mathcal{M} \models \exists(c,d) & \text{iff} & [\![c]\!] \cap [\![d]\!] \neq \emptyset \\ \mathcal{M} \models c(j) & \text{iff} & [\![c]\!]([\![j]\!]) \\ \mathcal{M} \models r(j,k) & \text{iff} & [\![r]\!]([\![j]\!], [\![k]\!]) \end{array}$$

If Γ is a set of formulas, we write $\mathcal{M} \models \Gamma$ if for all $\varphi \in \Gamma$, $\mathcal{M} \models \varphi$.

Example 10.1 For example, look back to Examples 5.1 and 6.2. This concerned the model $\mathcal{M}$ defined by $M = \{w, x, y, z\}$, $[\![p]\!] = \{w, x, y\}$, and with $[\![s]\!]$ shown below

$$\begin{tikzcd} w \arrow[r] \arrow[dr] & x \arrow[d] \arrow[dl] \\ y \arrow[u] \arrow[r, leftrightarrow] & z \arrow[loop right] \end{tikzcd}$$

Note at this point that $[\![\exists(\forall(p, \overline{s}), s)]\!] = \emptyset$. We also set $[\![j]\!] = w$ and $[\![k]\!] = x$. We get additional sentences true in $\mathcal{M}$ such as $s(j, k)$, $\overline{s}(k, j)$, and $\exists(\overline{p}, s)(k)$.

Here is a point that will be important later. For all terms c, $\mathcal{M} \models c(j)$ iff $\mathcal{M} \models c(k)$. (The easiest way to check this is to show that for all set terms c, $[\![c]\!]$ is one of the following four sets: $\emptyset, M, \{w, x, y\},$ or $\{z\}$.) However, $\mathcal{M} \models s(j, k)$ and $\mathcal{M} \models \overline{s}(k, j)$.

The satisfiability problem for the language is decidable for a very easy reason: the language $\mathcal{L}$ translates to the *two-variable fragment* FO^2 of first-order logic. (We shall see this shortly.) Thus we have the finite model property (by Mortimer [**?**]) and decidability of satisfiability in non-deterministic exponential time (Grädel et al [**?**]). It might therefore be interesting to ask whether the smaller fragment $\mathcal{L}$ is of a lower complexity. As it happens, it is. Pratt-Hartmann [**?**] showed that the satisfiability problem for a certain fragment $\mathcal{E}_2$ of FO^2 can be decided in ExpTime in the length of the input Γ, and his fragment was essentially the same as the one in this chapter.

The bar notation. We have already seen that our unary and binary atoms come with negative forms. We extend this notation to all sentences in the following ways: $\overline{\overline{p}} = p$, $\overline{\overline{s}} = s$, $\overline{\exists(l, r)} = \forall(l, \overline{r})$, $\overline{\forall(l, r)} = \exists(l, \overline{r})$, $\overline{\forall(c, d)} = \exists(c, \overline{d})$, $\overline{\exists(c, d)} = \forall(c, \overline{d})$, $\overline{c(j)} = \overline{c}(j)$, and $\overline{r(j, k)} = \overline{r}(j, k)$.

Translation of the syllogistic into $\mathcal{L}$. We indicate briefly a few translations to orient the reader. First, the classical syllogistic translates into $\mathcal{L}$:

All p are q	$\mapsto$	$\forall(p, q)$	No p are q	$\mapsto$	$\forall(p, \overline{q})$
Some p are q	$\mapsto$	$\exists(p, q)$	Some p aren't q	$\mapsto$	$\exists(p, \overline{q})$

We can also translate $\mathcal{L}$ to FO^2, the fragment of first order logic using only the variables x and w. We do this by mapping the set terms two ways, called $c \mapsto \varphi_{c,x}$ and $c \mapsto \varphi_{c,y}$. Here are the recursion equations for $c \mapsto \varphi_{c,x}$:

$$\begin{array}{lcllcl} p & \mapsto & P(x) & \qquad \forall(c, r) & \mapsto & (\forall y)(\varphi_{c,y}(y) \to r(x, y)) \\ \overline{p} & \mapsto & \neg P(x) & \qquad \exists(c, r) & \mapsto & (\exists y)(\varphi_{c,y}(y) \wedge r(x, y)) \end{array}$$

The equations for $c \mapsto \varphi_{c,y}$ are similar. Then the translation of the sentences into FO^2 follows easily.

We present our system in natural-deduction style in Figure 10.3. It makes use of *introduction* and *elimination* rules, and more critically of *variables*. For a textbook account of a proof system for first-order logic presented in this way, see van Dalen [**?**].

Expression	Variables	Syntax
individual variable	x, y	
individual term	t, u	$x \mid j$
general sentence	α	$\varphi \mid c(x) \mid r(x,y) \mid \bot$

Figure 10.2: Syntax of general sentences of $\mathcal{L}$, with φ ranging over sentences.

General sentences in this fragment are what usually are called *formulas*. We prefer to change the standard terminology to make the point that here, sentences are not built from formulas by quantification. In fact, sentences in our sense do not have variable occurrences. But general sentences do include *variables*. They are only used in our proof theory.

The syntax of general sentences is given in Figure 10.2. What we are calling *individual terms* are just variables and constant symbols. (There are no function symbols here.) Using terms allows us to shorten the statements of our rules, but this is the only reason to have terms.

An additional note: we don't need general sentences of the form $r(j,x)$ or $r(x,j)$. In larger fragments, we would expect to see general sentences of these forms, but our proof theory will not need these.

The bar notation, again. We have already seen the bar notation $\overline{c}$ for set terms c, and $\overline{\varphi}$ for sentences φ. We extend this to formulas $\overline{b(x)} = \overline{b}(x)$, $\overline{r(x,y)} = \overline{r}(x,y)$. We technically have a general sentence $\overline{\bot}$, but this plays no role in the proof theory.

We write $\Gamma \vdash \varphi$ if there is a proof tree conforming to the rules of the system with root labeled φ and whose axioms are labeled by elements of Γ. (Frequently we shall be sloppy about the labeling and just speak, e.g, of the root as if it *were* a sentence instead of being *labeled by* one.) Instead of giving a precise definition here, we shall content ourselves with a series of examples in Section 10.2 just below.

The system has two rules called $(\forall E)$, one for deriving general sentences of the form $c(x)$ or $c(j)$, and one for deriving general sentences $r(x,y)$ or $r(j,k)$. (Other rules are doubled as well, of course.) It surely looks like these should be unified, and the system would of course be more elegant if they were. But given the way we are presenting the syntax, there is no way to do this. That is, we do not have a concept of *substitution*, and so rules like $(\forall E)$ cannot be formulated in the usual way. Returning to the two rules with the same name, we could have chosen to use different names, say $(\forall E1)$ and $(\forall E2)$. But the result would have been a more cluttered notation, and it is always clear from context which rule is being used.

Although we are speaking of trees, we don't distinguish left from right. This is especially the case with the $(\exists E)$ rules, where the canceled hypotheses may occur in either order.

Side Conditions. As with every natural deduction system using variables, there are some side conditions which are needed in order to have a *sound* system.

$$\frac{c(t) \quad \forall(c,d)}{d(t)}\ \forall E \qquad\qquad \frac{c(u) \quad \forall(c,r)(t)}{r(t,u)}\ \forall E$$

$$\frac{c(t) \quad d(t)}{\exists(c,d)}\ \exists I \qquad\qquad \frac{r(t,u) \quad c(u)}{\exists(c,r)(t)}\ \exists I$$

$$\frac{\begin{matrix}[c(x)] \\ \vdots \\ d(x)\end{matrix}}{\forall(c,d)}\ \forall I \qquad\qquad \frac{\begin{matrix}[c(x)] \\ \vdots \\ r(t,x)\end{matrix}}{\forall(c,r)(t)}\ \forall I$$

$$\frac{\exists(c,d) \qquad \begin{matrix}[c(x)] \quad [d(x)] \\ \vdots \\ \alpha\end{matrix}}{\alpha}\ \exists E \qquad\qquad \frac{\exists(c,r)(t) \qquad \begin{matrix}[c(x)] \quad [r(t,x)] \\ \vdots \\ \alpha\end{matrix}}{\alpha}\ \exists E$$

$$\frac{\alpha \quad \overline{\alpha}}{\bot}\ \bot I \qquad\qquad \frac{\begin{matrix}[\overline{\varphi}] \\ \vdots \\ \bot\end{matrix}}{\varphi}\ \text{RAA}$$

Figure 10.3: Proof rules. See the text for the side conditions in the $(\forall I)$ and $(\exists E)$ rules.

In $(\forall I)$, x must not occur free in any uncanceled hypothesis. For example, in the version whose root is $\forall(c,d)$, one must cancel all occurrences of $c(x)$ in the leaves, and x must not appear free in any other leaf.

In $(\exists E)$, the variable x must not occur free in the conclusion α or in any uncanceled hypothesis in the subderivation of α.

In contrast to usual first-order natural deduction systems, there are *no side conditions* on the rules $(\forall E)$ and $(\exists I)$. The usual side conditions are phrased in terms of concepts such as *free substitution*, and the syntax here has no substitution to begin with. To be sure on this point, one should check the soundness result of Lemma 10.3.1.

Formal proofs in the Fitch style. The proof system in this chapter is presented in a standard Gentzen-style format. But it may easily be re-formatted to look more like a Fitch system, as we shall see in Example 10.3 and Figure 10.5. These examples might give the impression that we have merely re-presented Fitch-style natural deduction proofs. The difference is that our syntax is not a special case of the syntax of first-order logic. Corresponding to this, our proof rules our rather restrictive, and the system cannot be used for much of anything beyond the language $\mathcal{L}$. However, the fact that our Fitch-style proofs *look like* familiar formal proofs is a virtue: for example, it means that one could teach logic using this material.

10.2 Examples

We present a few examples of the proof system at work, along with comments pertaining to the side conditions. Many of these are taken from the proof system $\mathbf{R}^*$ for the language $\mathcal{RC}$ of [**?**]. That system $\mathbf{R}^*$ is among the strongest of the known syllogistic systems, and so it is of interest to check the current proof system is at least as strong.

Example 10.2 Here is a proof of the classical syllogism *Darii*: $\forall(b, d), \exists(c, b) \vdash \exists(c, d)$. First, in Fitch-style:

$$\begin{array}{r|ll}
1 & \forall(b,c) & \text{hyp} \\
2 & \exists(b,d) & \text{hyp} \\
3 & b(x) & \exists\text{E, } 2 \\
4 & d(x) & \exists\text{E, } 2 \\
5 & c(x) & \forall\text{E, } 1,\ 3 \\
6 & \exists(c,d) & \exists\text{I, } 4,\ 5
\end{array}$$

and then in natural deduction:

$$\dfrac{\exists(c,b) \qquad \dfrac{\dfrac{[b(x)]^1 \quad \forall(b,d)}{d(x)}\,\forall E \qquad [c(x)]^1}{\exists(c,d)}\,\exists I}{\exists(c,d)}\,\exists E^1$$

Example 10.3 Next we study a principle called (K) in Figure 6.2. Intuitively, if all watches are expensive items, then everyone who owns a watch owns an expensive item. The formal statement in our language is $\forall(c, d) \vdash \forall(\exists(c, r), \exists(d, r))$. See Figure 10.4. We present a Fitch-style proof on the left and the corresponding one in our formalism on the right. One aspect of the Fitch-style system is that $(\exists E)$ gives two lines; see lines 3 and 4 on the left in Figure 10.4.

Example 10.4 Here is an example of a derivation using (RAA). It shows $\forall(c, \overline{c}) \vdash \forall(d, \forall(c, r))$.

1		$\forall(c,d)$	hyp
2	x	$\exists(c,r)(x)$	hyp
3		$c(y)$	$\exists E$, 2
4		$r(x,y)$	$\exists E$, 2
5		$d(y)$	$\forall E$, 1, 3
6		$\exists(d,r)(x)$	$\exists I$, 4, 5
7		$\forall(\exists(c,r),\exists(d,r))$	$\forall I$, 1–6

$$\dfrac{\dfrac{[\exists(c,r)(x)]^2 \qquad \dfrac{[r(x,y)]^1 \qquad \dfrac{[c(y)]^1 \quad \forall(c,d)}{d(y)}\,\forall E}{\exists(d,r)(x)}\,\exists I}{\exists(d,r)(x)}\,\exists E^1}{\forall(\exists(c,r),\exists(d,r))}\,\forall I^2$$

Figure 10.4: Derivations in Example 10.3

$$\dfrac{[d(x)]^2 \qquad \dfrac{\dfrac{\dfrac{\dfrac{[c(y)]^1 \quad \forall(c,\overline{c})}{\overline{c}(y)}\,\forall E \qquad [c(y)]^1}{\bot}\,\bot I}{r(x,y)}\,\mathrm{RAA}}{\forall(c,r)(x)}\,\forall I^1}{\forall(d,\forall(c,r))}\,\forall I^2$$

Example 10.5 As in Lemma 6.3.1, we have the *rule of proof by cases*: If $\Gamma+\varphi\vdash\psi$ and $\Gamma+\overline{\varphi}\vdash\psi$, then $\Gamma\vdash\psi$.

10.3 Soundness

Before presenting a soundness result, it might be good to see an improper derivation. Here is one, purporting to infer some men see some men from some men see some women:

$$\dfrac{\exists(m,\exists(w,s)) \qquad \dfrac{[\exists(w,s)(x)]^2 \qquad \dfrac{\dfrac{[s(x,x)]^1 \quad [m(x)]^2}{\exists(s,m)(x)}\,\exists I \qquad [m(x)]^2}{\exists(m,\exists(m,s))}\,\exists I}{\exists(m,\exists(m,s))}\,\exists E^1}{\exists(m,\exists(m,s))}\,\exists E^2$$

The specific problem here is that when $[s(x,x)]$ is withdrawn in the application of $\exists I^1$, the variable x is free in the as-yet-uncanceled leaves labeled $m(x)$.

To state a result pertaining to the soundness of our system, we need to define the truth value of a general sentence under a variable assignment. First, a *variable assignment* in a model $\mathcal{M}$ is a function $v : V \to M$, where V is the set of variable symbols and M is the universe of $\mathcal{M}$. We need to define $\mathcal{M} \models \alpha[v]$ for general sentences α. If α is a sentence, then $\mathcal{M} \models \alpha[v]$ iff $\mathcal{M} \models \alpha$ in our earlier sense. If α is $b(x)$, then $\mathcal{M} \models \alpha[v]$ iff $[\![b]\!](v(x))$. If α is $r(x, y)$, then $\mathcal{M} \models \alpha[v]$ iff $[\![r]\!](v(x), v(y))$. If α is $\bot$, then $\mathcal{M} \not\models \bot$ for all models $\mathcal{M}$.

Lemma 10.3.1 *Let* Π *be any proof tree for this fragment all of whose nodes are labeled with* $\mathcal{L}$*-formulas, let* φ *be the root of* Π*, let* $\mathcal{M}$ *be a structure, let* $v : X \to M$ *be a variable assignment, and assume that for all uncanceled leaves* ψ *of* Π*,* $\mathcal{M} \models \psi[v]$*. Then also* $\mathcal{M} \models \varphi[v]$*.*

Proof By induction on Π. We shall only go into details concerning two cases. First, consider the case when the root of Π is

$$\dfrac{\exists(c,r)(t) \qquad \begin{matrix}[c(x)] \quad [r(t,x)] \\ \vdots \\ \alpha\end{matrix}}{\alpha}\ \exists E$$

To simplify matters further, let us assume that t is a variable. Let v be a variable assignment making true all of the leaves of the tree, except possibly $c(x)$ and $r(t, x)$. By induction hypothesis, $\mathcal{M} \models \exists(c, r)(t)[v]$. Let $a \in A$ witness this assertion. In the obvious notation, $[\![c]\!](a)$ and $[\![r]\!](t^{\mathcal{M},v}, a)$. Let w be the same variable assignment as v, except that $w(x) = a$. Then since x is not free in any leaves except those labeled $c(x)$ and $r(t, x)$, we have $\mathcal{M} \models \psi[w]$ for all those ψ. And so $\mathcal{M} \models \alpha[w]$, using the induction hypothesis applied to the subtree on the right. And since x is not free in the conclusion α, we also have $\mathcal{M} \models \alpha[v]$, as desired.

Second, let us consider the case when the root is

$$\dfrac{c(y) \quad \forall(c,r)(x)}{r(x,y)}\ \forall E$$

(That is, we are considering an instance of $(\forall E)$ when the terms t and u are variables.) The variables x and y might well be the same. Let $\mathcal{M}$ be a structure, and v be a variable assignment making true the leaves of the tree. By induction hypothesis, $[\![c]\!](v(y))$ and also $[\![r]\!](v(x), m))$ for all $m \in [\![c]\!]$. In particular, $[\![r]\!](v(x), v(y))$.

The remaining cases are similar. $\dashv$

10.4 The Henkin property

The completeness of the logic parallels the Henkin-style completeness result for first-order logic. Given a consistent theory Γ, we get a model of Γ in the following way: (1) take the underlying language $\mathcal{L}$, add constant symbols to the language to witness existential sentences; (2) extend Γ to a maximal consistent set in the larger language; and then (3) use the set of constant symbols as the carrier of a model in a canonical way. In the setting of this chapter, the work is in some ways easier than in the standard setting, and in some ways harder. There are more details to check, since the language has more basic constructs. But one doesn't need to take a quotient by equivalence classes, and in other ways the work here is easier.

Given two languages $\mathcal{L}$ and $\mathcal{L}'$, we say that $\mathcal{L}' \supseteq \mathcal{L}$ if every symbol (of any type) in $\mathcal{L}$ is also a symbol (of the same type) in $\mathcal{L}'$. In this chapter, the main case is when $\mathbf{P}(\mathcal{L}) = \mathbf{P}(\mathcal{L}')$, $\mathbf{R}(\mathcal{L}) = \mathbf{R}(\mathcal{L}')$, and $\mathbf{K}(\mathcal{L}) \subseteq \mathbf{K}(\mathcal{L}')$; that is, $\mathcal{L}'$ arises by adding constants to $\mathcal{L}$.

A *theory* in a language is just a set of sentences in it. Given a theory Γ in a language $\mathcal{L}$, and a theory Γ^* in an extension $\mathcal{L}' \supseteq \mathcal{L}$, we say that Γ^* is a *conservative extension* of Γ if for every $\varphi \in \mathcal{L}$, if $\Gamma^* \vdash \varphi$, then $\Gamma \vdash \varphi$.

Lemma 10.4.1 *Let Γ be a consistent $\mathcal{L}$-theory, and let $j \notin \mathbf{K}(\mathcal{L})$.*

(i) If $\exists(c, d) \in \Gamma$, then $\Gamma + c(j) + d(j)$ is a conservative extension of Γ.

(ii) If $\exists(c, r)(j) \in \Gamma$, then $\Gamma + r(j, k) + c(k)$ is a conservative extension of Γ.

Proof For (1), suppose that Γ contains $\exists(c, d)$ and that $\Gamma + c(j) + d(j) \vdash \varphi$. Let Π be a derivation tree. Replace the constant j by an individual variable x which does not occur in Π. The result is still a derivation tree, except that the leaves are not labeled by *sentences*. (The reason is that our proof system has no rules specifically for constants, only for terms which might be constants and also might be individual variables.) Call the resulting tree Π'. Now the following proof tree shows that $\Gamma \vdash \varphi$:

$$\dfrac{\exists(c,d) \qquad \begin{array}{c} [c(x)] \quad [d(x)] \\ \vdots \\ \varphi \end{array}}{\varphi}\ \exists E$$

The subtree on the right is Π'. The point is that the occurrences of $c(x)$ and $d(x)$ have been canceled by the use of $\exists E$ at the root.

This completes the proof of the first assertion, and the proof of the second is similar. $\dashv$

Definition An $\mathcal{L}$-*theory* Γ has the *Henkin property* if the following hold:

(i) If $\exists(c, d) \in \Gamma$, then for some constant j, $c(j)$ and $d(j)$ belong to Γ.

(ii) If r is a literal of $\mathcal{L}$ and $\exists(c, r)(j) \in \Gamma$, then for some constant k, $r(j, k)$ and $c(k)$ belong to Γ.

Lemma 10.4.2 *Let Γ be a consistent $\mathcal{L}$-theory. Then there is some $\mathcal{L}^* \supset \mathcal{L}$ and some $\mathcal{L}^*$-theory Γ^* such that Γ^* is a maximal consistent theory with the Henkin property. Moreover, if $s \in \mathbf{R}(\mathcal{L})$, $j \in \mathbf{K}(\mathcal{L}^*)$ and $k \in \mathbf{K}(\mathcal{L})$, and if $s(j,k) \in \Gamma^*$, then $j \in \mathbf{K}(\mathcal{L})$.*

Proof This is a routine argument, using Lemma 10.4.1. One dovetails the addition of constants which is needed for the Henkin property together with the addition of sentences needed to insure maximal consistency. The formal details would use Lemma 10.4.1 for steps of the first kind, and for the second kind we need to know that if Γ is consistent, then for all φ, either $\Gamma + \varphi$ or $\Gamma + \overline{\varphi}$ is consistent. This follows from the derivable rule of proof by cases; see Lemma 6.3.1 in Section 10.2. $\dashv$

The last point in Lemma 10.4.2 states a technical property that will be useful in Section 10.8.

It might be worthwhile noting that the extensions produced by Lemma 10.4.2 add *infinitely many constants* to the language.

10.5 Completeness via canonical models

In this section, fix a language $\mathcal{L}$ and a maximal consistent Henkin $\mathcal{L}$-theory Γ. We construct a *canonical model* $\mathcal{M} = \mathcal{M}(\Gamma)$ as follows: $M = \mathbf{K}(\mathcal{L})$; $[\![p]\!](j)$ iff $p(j) \in \Gamma$; $[\![s]\!](j,k)$ iff $s(j,k) \in \Gamma$; and $[\![j]\!] = j$. That is, we take the constant symbols of the language to be the points of the model, and the interpretations of the atoms are the natural ones. Each constant symbol is interpreted by itself.

Lemma 10.5.1 *For all set terms c, $[\![c]\!] = \{j : c(j) \in \Gamma\}$.*

Proof By induction on c. The base case of unary atoms p is by definition of $\mathcal{M}$.

Before we turn to the induction proper, here is a preliminary point. Assuming that $[\![c]\!] = \{j : c(j) \in \Gamma\}$, we check that $[\![\overline{c}]\!] = \{j : \overline{c}(j) \in \Gamma\}$:

$$j \in [\![\overline{c}]\!] \quad \text{iff} \quad j \notin [\![c]\!] \quad \text{iff} \quad c(j) \notin \Gamma \quad \text{iff} \quad \overline{c}(j) \in \Gamma.$$

The last point uses the maximal consistency of Γ.

Turning to the inductive steps, assume our result for c; we establish it for $\forall(c,s)$ and $\exists(c,s)$; it then follows from the preliminary point that we have the same fact for $\forall(c,\overline{s})$ and $\exists(c,\overline{s})$.

Let $j \in [\![\forall(c,s)]\!]$. We claim that $\forall(c,s)(j) \in \Gamma$. For if not, then $\exists(c,\overline{s})(j) \in \Gamma$. By the Henkin property, let k be such that Γ contains $c(k)$ and $\overline{s}(j,k)$. By the induction hypothesis, $k \in [\![c]\!]$, and by the definition of $\mathcal{M}$, $[\![s]\!](j,k)$ is false. Thus $j \notin [\![\forall(c,s)]\!]$. This is a contradiction.

In the other direction, assume that $\forall(c,s)(j) \in \Gamma$; this time we claim that $j \in [\![\forall(c,s)]\!]$. Let $k \in [\![c]\!]$. By induction hypothesis, Γ contains $c(k)$. By $(\forall E)$, we see that $\Gamma \vdash s(j,k)$. Hence Γ contains $s(j,k)$. So in $\mathcal{M}$, $[\![s]\!](j,k)$. Since k was arbitrary, we see that indeed $j \in [\![\forall(c,s)]\!]$.

The other induction step is for $\exists(c,s)$. Let $j \in [\![\exists(c,s)]\!]$. We thus have some $k \in [\![c]\!]$ such that $[\![s]\!](j,k)$. That is, $s(j,k) \in \Gamma$. Using $(\exists I)$, we have $\Gamma \vdash \exists(c,s)(j)$; from this we see that $\exists(c,s)(j) \in \Gamma$, as desired.

Finally, assume that $\exists(c,s)(j) \in \Gamma$. By the Henkin condition, let k be such that Γ contains $c(k)$ and $s(j,k)$. Using the derivation above, we have the desired conclusion that $j \in [\![\exists(c,s)]\!]$.

This concludes the proof. $\dashv$

Lemma 10.5.2 $\mathcal{M} \models \Gamma$.

Proof We check the sentence types in turn. Throughout the proof, we shall use Lemma 10.5.1 without mention.

First, let Γ contain the sentence $\forall(c,d)$. Let $j \in [\![c]\!]$, so that $c(j) \in \Gamma$. We have $d(j) \in \Gamma$ using $(\forall E)$. This for all j shows that $\mathcal{M} \models \forall(c,d)$.

Second, let $\exists(c,d) \in \Gamma$. By the Henkin condition, let j be such that both $c(j)$ and $d(j)$ belong to Γ. This element j shows that $[\![c]\!] \cap [\![d]\!] \neq \emptyset$. That is, $\mathcal{M} \models \exists(c,d)$.

Continuing, consider a sentence $c(j) \in \Gamma$. Then $j \in [\![c]\!]$, so that $\mathcal{M} \models c(j)$.

Finally, the case of sentences $r(j,k) \in \Gamma$ is immediate from the structure of the model. $\dashv$

Theorem 10.5.3 *If* $\Gamma \models \varphi$, *then* $\Gamma \vdash \varphi$.

Proof We rehearse the standard argument. Due to the classical negation, we need only show that consistent sets Γ are satisfiable. Let $\mathcal{L}$ be the language of Γ, Let $\mathcal{L}' \supseteq \mathcal{L}$ be an extension of $\mathcal{L}$, and let $\Gamma^* \supseteq \Gamma$ be a maximal consistent theory in $\mathcal{L}'$ with the Henkin property (see Lemma 10.4.2). Consider the canonical model $\mathcal{M}(\Gamma^*)$ as defined in this section. By Lemma 10.5.2, $\mathcal{M}(\Gamma^*) \models \Gamma^*$. Thus Γ^* is satisfiable, and hence so is Γ. $\dashv$

10.6 The finite model property

Let Γ be a consistent finite theory in some language $\mathcal{L}$. As we now know, Γ has a model. Specifically, we have seen that there is some $\Gamma^* \supseteq \Gamma$ which is a maximal consistent theory with the Henkin property in an extended language $\mathcal{L}^* \supseteq \mathcal{L}$. Then we may take the set of constant symbols of $\mathcal{L}^*$ to be the carrier of a model of Γ^*, hence of Γ. The model obtained in this way is infinite. It is of interest to build a finite model, so in this section Γ must be finite. The easiest way to see that Γ has a finite model is to recall that our overall language is a sub-language of the two variable fragment FO^2 of first-order logic. And FO^2 has the finite model property by Mortimer's Theorem [?].

However, it is possible to give a direct argument for the finite model property, along the lines of filtration in modal logic (but with some differences). We sketch the result here because we shall use the same method in Section 10.8 below to prove a finite model property for our second logical system $\mathcal{L}(adj)$ with respect to its natural semantics; that result does not follow from others in the literature.

Let $\mathcal{M} = \mathcal{M}(\Gamma^*)$ be the canonical model as defined in Section 10.5. Let $Sub(\Gamma)$ be the collection of set terms occurring in any sentence in the original finite theory Γ. So $Sub(\Gamma)$ is finite, and if $\forall(c, r) \in Sub(\Gamma)$ or $\exists(c, r) \in Sub(\Gamma)$, then also $c \in Sub(\Gamma)$. For constant symbols j and k of $\mathcal{L}^*$, write $j \equiv k$ iff the following conditions hold:

(i) If either j or k is a constant of $\mathcal{L}$, then $k = j$.

(ii) For all $c \in Sub(\Gamma)$, $c(j) \in \Gamma$ iff $c(k) \in \Gamma$.

check the reference

Remark The equivalence relation $\equiv$ may be defined on any structure. It is not necessarily a congruence, as Example 5.1 shows. Specifically, we had constant symbols j and k such that $j \equiv k$, and yet in our structure $s(j, k)$ and $\overline{s}(k, j)$. In the case of $\mathcal{M}(\Gamma^*)$, we have no reason to think that $\equiv$ is a congruence. That is, the construction in Section 10.5 did not arrange for this.

Let $N = \{[k] : k \in \mathbf{K}(\mathcal{L})\} \times \{\forall, \exists\}$. (We use $\forall$ and $\exists$ as tags to give two copies of the quotient $\mathbf{K}/\!\equiv$.) We endow N with an $\mathcal{L}$-structure as follows:

(i) $[\![p]\!] = \{([j], Q) : p(j) \in \Gamma^* \text{ and } Q \in \{\forall, \exists\}\}$.

(ii) $[\![s]\!](([j], Q), ([k], Q'))$ iff one of the following two conditions holds:
   a) There is a set term c such that Γ^* contains $c(k)$ and $\forall(c, s)(j)$.
   b) $Q' = \exists$, and for some $j_* \equiv j$ and $k_* \equiv k$, Γ^* contains $s(j_*, k_*)$.

(iii) For a constant j of $\mathcal{L}$, $[\![j]\!] = ([j], \exists)$. (Of course, $[j]$ is the singleton set $\{j\}$.)

Before going on, we note that the first of the two alternatives in the definition of $[\![s]\!](([j], Q), ([k], Q'))$ is independent of the choice of representatives of equivalence classes. And clearly so is the second alternative.

We shall write $\mathcal{N}$ for the resulting $\mathcal{L}$-structure, hiding the dependence on Γ and Γ^*.

Lemma 10.6.1 *For all $c \in \mathrm{Sub}(\Gamma)$, $[\![c]\!] = \{([j], Q) : c(j) \in \Gamma^*$ and $Q \in \{\forall, \exists\}\}$.*

Proof By induction on set terms c. We are not going to present any of the details here because in Lemma 10.8.3 below, we shall see all the details on a more involved result. $\dashv$

Lemma 10.6.2 *$\mathcal{N} \models \Gamma$.*

Proof Again we only highlight a few details, since the full account is similar to what we saw in Lemma 10.5.2, and to what we shall see in Lemma 10.8.4. One would check the sentence types in turn, using Lemma 10.6.1 frequently. We want to go into details concerning sentences in Γ of the form $s(j, k)$ or $\overline{s}(j, k)$. Recall that we are dealing in this result with sentences of $\mathcal{L}$, and so j and k are constant symbols of that language. Also recall that $[\![j]\!] = ([j], \exists)$, and similarly for k.

First, consider sentences in Γ of the form $s(j,k)$. By the definition of $[\![s]\!]$, we have

$$[\![s]\!](([j],\exists),([k],\exists)).$$

By the way binary atoms and constants are interpreted in $\mathcal{N}$, we have $\mathcal{N} \models s(j,k)$, as desired.

We conclude with the consideration of a sentence in Γ of the form $\overline{s}(j,k)$. We wish to show that $\mathcal{N} \models \overline{s}(j,k)$. Suppose towards a contradiction that $\mathcal{N} \models s(j,k)$. Then we have $[\![s]\!](([j],\exists),([k],\exists))$. There are two possibilities, corresponding to the alternatives in the semantics of s. The first is when there is a set term c such that Γ^* contains $c(k)$ and $\forall(c,s)(j)$. By $(\forall E)$, Γ^* then contains $s(j,k)$. But recall that Γ contains $\overline{s}(j,k)$. So in this alternative, $\Gamma^* \supseteq \Gamma$ is inconsistent. In the second alternative, there are $j_* \equiv j$ and $k_* \equiv k$ such that $s(j_*,k_*) \in \Gamma^*$. But recall that the equivalence classes of constant symbols from the base language $\mathcal{L}$ are singletons. Thus in this alternative, $j_* = j$ and $k_* = k$; hence $s(j,k) \in \Gamma^*$. But then again Γ^* is inconsistent, a contradiction. $\dashv$

Theorem 10.6.3 (Finite Model Property) *If Γ is consistent, then Γ has a model of size at most 2^{2n}, where n is the number of set terms in Γ.*

Complexity notes. Theorem 10.6.3 implies that the satisfiability problem for our language is in NExpTime. We can improve this to an ExpTime-completeness result by quoting the work of others. Pratt-Hartmann [**?**] defines a logic called $\mathcal{E}_2$ and showed that the complexity of its satisfiability problem is ExpTime-complete. $\mathcal{E}_2$ corresponds to a fragment of first-order logic, and it is somewhat bigger than the language $\mathcal{L}$. (It would correspond to adding converses to the binary atoms in $\mathcal{L}$, as we mention at the very end of this chapter.) Since satisfiability for $\mathcal{E}_2$ is ExpTime-complete, the same problem for $\mathcal{L}$ is in ExpTime.

A different way to obtain this upper bound is via the embedding into Boolean modal logic which we saw in Section **??**. For this, see Theorem 7 of Lutz and Sattler [**?**]. We shall use an extension of that result below in connection with an extension $\mathcal{L}(adj)$ of $\mathcal{L}$.

The ExpTime-hardness for $\mathcal{L}$ follows from Lemma 6.1 in [**?**]. That result dealt with a language called $\mathcal{R}^\dagger$, and $\mathcal{R}^\dagger$ is a sub-language of $\mathcal{L}$.

10.7 Adding transitivity: $\mathcal{L}(adj)$

Before going further, let us briefly recapitulate the overall problem of this chapter and point out where we are and what remains to be done. We aim to formalize a fragment of first-order logic in which one may represent arguments as complex as that in (10.1) in the Introduction. We are especially interested in decidable systems, and so the systems must be weaker than first-order logic. We presented in Section 10 a language $\mathcal{L}$ and a proof system for it. Validity in the logic cannot be captured by a purely syllogistic proof system, and so our proof system uses variables. But the use is very special and restricted. The proof system is complete and decidable in exponential time. To our knowledge, it is the first system with these properties.

There are a number of ways in which one can go further. In this section, we want to explore one such way, connected to the example in (10.1) below:

$$\begin{array}{l} \textsf{Every sweet fruit is bigger than every ripe fruit} \\ \textsf{Every pineapple is bigger than every kumquat} \\ \textsf{Every non-pineapple is bigger than every unripe fruit} \\ \hline \textsf{Every fruit bigger than some sweet fruit is bigger than every kumquat} \end{array} \tag{10.1}$$

One key feature of this example is that comparative adjectives such as bigger than are transitive. This is true for all comparative adjectives.

We extend our language $\mathcal{L}$ to a language $\mathcal{L}(adj)$ by taking a basic set $\mathbf{A}$ of comparative adjective phrases in the base. The proof system simply extends the one we have already seen with a rule corresponding to the transitivity of comparatives. Our completeness result, Theorem 10.5.3, extends to the new setting. The next section does this. The decidability of the language is a more delicate matter than before, since it does not follow from Mortimer's Theorem [**?**] on the finite model property for FO^2. Indeed, adding transitivity statements to FO^2 renders the logic undecidable, as shown in Grädel, Otto, and Rosen [**?**]. Instead, one could use Theorem 12 of Lutz and Sattler [**?**] on the decidability of a variant on Boolean modal logic in which some of the relations are taken to be transitive. This would indeed give the ExpTime-completeness of $\mathcal{L}(adj)$ with our semantics. However, we have decided to present a direct proof for several reasons. First, Lutz and Sattler's result does not give a finite model property, and our result does do this. Second, our argument is shorter. Finally, our treatment connects to modal filtration arguments and is therefore different; [**?**] uses automata on infinite trees and is based on Vardi and Wolper [**?**].

I do not wish to treat the transitivity of comparison with adjectives as an enthymeme (missing premise) because the transitivity seems more fundamental, more 'logical' somehow. Hence it should be treated on a deeper level. The decidability considerations give a supporting argument: if we took the transitivity to be a *meaning postulate*, then it would seem that the underlying language would have to be rich enough to state transitivity. This requires three universal quantifiers. For other reasons, we want our languages to be closed under negation. It thus seems very likely that any logical system with these properties is going to be undecidable. The upshot is a system in which the transitivity turns out to be a *proof postulate* rather than a *meaning postulate*. We turn to the system itself.

Syntax and semantics. We start with four pairwise disjoint sets $\mathbf{A}$ (for *comparative adjective phrases*) and the three that we saw before: $\mathbf{P}$, $\mathbf{R}$, and $\mathbf{K}$. We use a as a variable to range over $\mathbf{A}$ in our statement of the syntax and the rules.

For the syntax, we take elements $a \in \mathbf{A}$ to be binary atoms, just as the elements $s \in \mathbf{R}$ are. Thus, the binary literals are the expressions of the form s, $\overline{s}$, a, or $\overline{a}$.

The syntax is the same as before, except that we allow the binary atoms to be elements of $\mathbf{A}$ in addition to elements of $\mathbf{R}$. So in a sense, we have the same syntax is before, except that some of the binary atoms are taken to render transitive verbs, and some are taken to render comparative adjective phrases. The only difference is in the semantics. Here, we require that (in every model $\mathcal{M}$) for an adjective $a \in \mathbf{A}$, $[\![a]\!]$ must be a transitive relation.

Proof system. We adopt the same proof system as in Figure 10.3, but with one addition. This addition is the rule for transitivity:

$$\frac{a(t_1, t_2) \quad a(t_2, t_3)}{a(t_1, t_3)}\ \text{trans}$$

This rule is added for all $a \in \mathbf{A}$.

Example 10.6 We have seen an informal example in (10.1) at the beginning of this chapter. At this point, we can check that our system does indeed have a derivation corresponding to this. We need to check that $\Gamma \vdash \varphi$, where Γ contains

$$\forall(\mathsf{sw}, \forall(\mathsf{ripe}, \mathsf{bigger})), \forall(\mathsf{pineapple}, \forall(\mathsf{kq}, \mathsf{bigger})), \forall(\overline{\mathsf{pineapple}}, \forall(\overline{\mathsf{ripe}}, \mathsf{bigger})),$$

and φ is

$$\forall(\exists(\mathsf{sw}, \mathsf{bigger}), \forall(\mathsf{kq}, \mathsf{bigger})).$$

(We are going to use kq as an abbreviation of kumquat for typographical convenience, and similarly for sw and sweet.)

Example 10.7 The example at the beginning of this chapter cannot be formalized in this fragment because the correct reasoning uses the transitivity of is bigger than. However, we can prove a result which may itself be used in a formal proof of (10.1):

$$\frac{\begin{array}{l}\text{Every sweet fruit is bigger than every ripe fruit}\\ \text{Every pineapple is bigger than every kumquat}\\ \text{Every non-pineapple is bigger than every unripe fruit}\end{array}}{\text{Every sweet fruit is bigger than every kumquat}} \tag{10.2}$$

To discuss this, we take the set $\mathbf{P}$ of unary atoms to be

$$\mathbf{P} \quad = \quad \{\mathsf{sweet}, \mathsf{ripe}, \mathsf{pineapple}, \mathsf{kumquat}\}.$$

We also take $\mathbf{R} = \{\mathsf{bigger}\}$ and $\mathbf{K} = \emptyset$. Figure 10.5 contains a derivation showing (10.2), done in the manner of Fitch [**?**]. The main way in which we have bent the English in the direction of our formalism is to use the bar notation on the nouns. The main reason for presenting the derivation as a Fitch diagram is that the derivation given as a tree

1	Every sweet fruit is bigger than every ripe fruit	hyp
2	Every pineapple is bigger than every kumquat	hyp
3	Every $\overline{\text{pineapple}}$ is bigger than every $\overline{\text{ripe fruit}}$	hyp
4	x | x is a sweet fruit	hyp
5	x is bigger than every ripe fruit	$\forall E$, 1, 4
6	x is a pineapple	hyp
7	x is bigger than every kumquat	$\forall E$, 2, 6
8	x is a $\overline{\text{pineapple}}$	hyp
9	x is bigger than every $\overline{\text{ripe fruit}}$	$\forall E$, 3, 8
10	y | y is a kumquat	hyp
11	y is a ripe fruit	hyp
12	x is bigger than y	$\forall E$, 5, 11
13	y is a $\overline{\text{ripe fruit}}$	hyp
14	x is bigger than y	$\forall E$, 9, 13
15	x is bigger than y	cases, 13–14, 11–12
16	x is bigger than every kumquat	$\forall I$, 10–15
17	x is bigger than every kumquat	cases, 6–7, 8–16
18	Every sweet fruit is bigger than every kumquat	$\forall I$, 4–17

Figure 10.5: A derivation corresponding to the argument in (10.2).

(as demanded by our definitions) would not fit on a page. This is because the cases rule is not a first-class rule in the system, it is a derived rule (see Lemma 6.3.1). Our Fitch diagram pretends that the system has a rule of cases. Another reason to present the derivation as in Figure 10.5 is to make the point that the treatment in this chapter is a beginning of a formalization of the work that Fitch was doing.

check all the references on this

In Figure 10.5, we see that $\Gamma \vdash \forall(\mathsf{sweet}, \forall(\mathsf{kq}, \mathsf{bigger}))$. (The a derivation was presented using a format which could be converted to our official format of natural deduction trees.) That work used $\mathbf{R} = \{\mathsf{bigger}\}$, but here we want $\mathbf{R} = \emptyset$ and $\mathbf{A} = \{\mathsf{bigger}\}$. The same derivation works, of course. Transitivity enables us to obtain a derivation for (10.1):

$$
\begin{prooftree}
\AxiomC{$[\exists(\mathsf{sw}, \mathsf{bigger})(x)]^3$}
\AxiomC{$[\mathsf{bigger}(x,y)]^2$}
\AxiomC{$[\mathsf{kq}(z)]^1$}
\AxiomC{$[\mathsf{sw}(y)]^2$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\forall(\mathsf{sw}, \forall(\mathsf{kq}, \mathsf{bigger}))$}
\RightLabel{$\forall E$}
\BinaryInfC{$\forall(\mathsf{kq}, \mathsf{bigger})(y)$}
\RightLabel{$\forall E$}
\BinaryInfC{$\mathsf{bigger}(y,z)$}
\RightLabel{trans}
\BinaryInfC{$\mathsf{bigger}(x,z)$}
\RightLabel{$\forall I^1$}
\UnaryInfC{$\forall(\mathsf{kq}, \mathsf{bigger})(x)$}
\RightLabel{$\exists E^2$}
\BinaryInfC{$\forall(\mathsf{kq}, \mathsf{bigger})(x)$}
\RightLabel{$\forall I^3$}
\UnaryInfC{$\forall(\exists(\mathsf{sw}, \mathsf{bigger}), \forall(\mathsf{kq}, \mathsf{bigger}))$}
\end{prooftree}
$$

Adding the transitivity rule gives a sound and complete proof system for the semantic consequence relation $\Gamma \models \varphi$. The soundness is easy, and so we only sketch the completeness. We must show that a set Γ which is consistent in the new logic has a transitive model. The canonical model $\mathcal{M}(\Gamma)$ as defined in Section 10.5 is automatically transitive; this is immediate from the transitivity rule. And as we know, it satisfies Γ.

10.8 $\mathcal{L}(adj)$ has the finite model property

Our final result is that $\mathcal{L}(adj)$ has the finite model property. We extend the work in Section 10.6. The inspiration for our definitions comes from the technique of *filtration* in modal logic, but we shall not refer explicitly to this area.

We again assume that Γ is consistent, and Γ^* has the properties of Lemma 10.4.2.

Definition For $a \in \mathbf{A}$, we say that j *reaches* k (*by a chain of* $\equiv$ *and* a *statements*) if there is a sequence

$$
j = j_0 \;\equiv\; k_0, \quad j_1 \;\equiv\; k_1, \quad \ldots, \quad j_n \;\equiv\; k_n = k \tag{10.3}
$$

such that $n \geq 1$, and Γ^* contains $a(k_0, j_1)$, …, $a(k_{n-1}, j_n)$.

Lemma 10.8.1 *Assume that j reaches k by a chain of $\equiv$ and a statements.*

(i) If $c(k) \in \Gamma^*$, then Γ^* contains $\exists(c, a)(j)$.

(ii) If $j, k \in \mathbf{K}(\mathcal{L})$, then Γ^* contains $a(j, k)$.

Proof By induction on $n \geq 1$ in (10.3). For $n = 1$, we have essentially seen the argument as a step in Lemma 10.6.1. Here it is again. Since $c(k_1)$ and $j_1 \equiv k_1$, we see that $c(j_1)$. Together with $s(k_0, j_1)$, we have $\exists(c, a)(k_0)$. And as $j_0 \equiv k_0$, we see that $\exists(c, a)(j_0)$.

Assume our result for n, and now consider a chain as in (10.3) of length $n+1$. The induction hypothesis applies to

$$j = j_1 \equiv k_1, \quad j_2 \equiv k_2, \quad \ldots, \quad j_{n+1} \equiv k_{n+1} = k$$

and so we have $\exists(c,a)(j_1)$. Since $a(k_0, j_1)$, we easily have $\exists(c,a)(k_0)$ by transitivity. And as $j_0 \equiv k_0$, we have $\exists(c,a)(j_0)$.

The second assertion is also proved by induction on $n \geq 1$. For $n = 1$, we have $j = j_0 \equiv k_0$, Γ^* contains $a(k_0, j_1)$; and $j_1 \equiv k_1 = k$. Then since the $\equiv$ is the identity on $\mathbf{K}(\mathcal{L})$, $j = j_0 = k_0$, and $j_1 = k_1 = k$. Hence Γ^* contains $s(j,k)$. Assuming our result for n, we again consider a chain as in (10.3) of length $n+1$. Just as before, $j = j_0 = k_0$, and so Γ^* contains $a(j, j_1)$. By induction hypothesis, Γ^* contains $a(j_1, k)$. By transitivity, Γ^* contains $a(j,k)$. ⊣

Is 'itemize' right for this?

We endow N with an $\mathcal{L}$-structure as follows:

$[\![p]\!] = \{([j], Q) : p(j) \in \Gamma^* \text{ and } Q \in \{\forall, \exists\}\}$.

$[\![s]\!](([j], Q), ([k], Q'))$ iff one of the following two conditions holds:

(i) There is a set term c such that Γ^* contains $c(k)$ and $\forall(c,s)(j)$.

(ii) $Q' = \exists$, and for some $j_* \equiv j$ and $k_* \equiv k$, Γ^* contains $s(j_*, k_*)$.

$[\![a]\!](([j], Q), ([k], Q'))$ iff

(i) If $\forall(c,a)(k) \in \Gamma^*$, then also $\forall(c,a)(j) \in \Gamma^*$.

(ii) In addition, either (a) or (b) below holds:

a) There is a set term c such that Γ^* contains $c(k)$ and $\forall(c,a)(j)$.

b) $Q' = \exists$, and j reaches k by a chain of $\equiv$ and a statements.

(Notice that this definition is independent of the representatives in $[j]$ and $[k]$.)

For a constant j of $\mathcal{L}$, $[\![j]\!] = ([j], \exists)$.

Once again, we suppress Γ and Γ^* and simply write $\mathcal{N}$ for the resulting $\mathcal{L}$-structure.

Lemma 10.8.2 *For $a \in \mathbf{A}$, each relation $[\![a]\!]$ is transitive in $\mathcal{N}$.*

???

Proof In this proof and the next , we are going to use l to stand for a constant symbol, even though earlier in the chapter we used it for a literal. Assume that

$$([j], Q) \; [\![a]\!] \; ([k], Q') \; [\![a]\!] \; ([l], Q''). \tag{10.4}$$

Clearly we have the first requirement concerning $[\![a]\!]$: if $\forall(c,a)(l) \in \Gamma^*$, then also $\forall(c,a)(j) \in \Gamma^*$.

We have four cases, depending on the reasons for the two assertions in (10.4).

<u>Case 1</u> There is a set term b such that Γ^* contains $b(k)$ and $\forall(b,a)(j)$, and there is also a set term c such that Γ^* contains $c(l)$ and $\forall(c,a)(k)$. By (1), Γ^* contains $c(l)$ and $\forall(c,a)(j)$. And so we have requirement (2a) concerning $[\![a]\!]$ for $([j], Q)$ and $([l], Q'')$.

Case 2 There is a set term b such that Γ^* contains $b(k)$ and $\forall(b,a)(j)$, and k reaches l. Note that $a(j,k)$. So j reaches l.

Case 3 j reaches k by a chain of $\equiv$ and a statements, and there is a set term c such that Γ^* contains $c(l)$ and $\forall(c,a)(k)$. Then $a(k,l)$. And so j reaches l.

Case 4 j reaches k, and k reaches l. Then concatenating the chains shows that j reaches l. $\dashv$

Lemma 10.8.3 *For all* $c \in \mathrm{Sub}(\Gamma)$, $[\![c]\!] = \{([j],Q) : c(j) \in \Gamma^* \text{ and } Q \in \{\forall,\exists\}\}$.

Proof We argue by induction on c. Much of the proof is as in Lemma 10.6.1, For c a unary atom, the result is obvious. Also, assuming that $[\![c]\!] = \{([j],Q) : c(j) \in \Gamma^*\}$ we easily have the same result for $\overline{c}$ using the maximal consistency of Γ^*:

$$([j],Q) \in [\![\overline{c}]\!] \quad \text{iff} \quad ([j],Q) \notin [\![c]\!] \quad \text{iff} \quad c(j) \notin \Gamma^* \quad \text{iff} \quad \overline{c}(j) \in \Gamma^*.$$

Assume about c that if $c \in Sub(\Gamma)$, then $[\![c]\!] = \{([j],Q) : c(j) \in \Gamma^*\}$. In view of what we just saw, we only need to check the same result for $\forall(c,s)$, $\exists(c,s)$, $\forall(c,a)$, and $\exists(c,\overline{a})$.

$\forall(c,s)$ Suppose that $\forall(c,s) \in Sub(\Gamma)$, so that $c \in Sub(\Gamma)$ as well. We prove that

$$[\![\forall(c,s)]\!] \quad = \quad \{([j],Q) : \forall(c,s)(j) \in \Gamma^*\}.$$

Let $([j],Q) \in [\![\forall(c,s)]\!]$. We shall show that $\forall(c,s)(j) \in \Gamma^*$. If not, then by maximal consistency, $\exists(c,\overline{s})(j) \in \Gamma^*$. By the Henkin property, let k be such that Γ^* contains $c(k)$ and $\overline{s}(j,k)$. By induction hypothesis, $([k],\forall) \in [\![c]\!]$. And so $([j],\forall)[\![s]\!]([k],\forall)$. Thus there is a set term b such that Γ^* contains $b(k)$ and $\forall(b,s)(j)$. From these, Γ^* contains $s(j,k)$. And thus Γ^* is inconsistent. This contradiction shows that indeed $\forall(c,s)(j) \in \Gamma^*$.

In the other direction, suppose that $([j],Q)$ is such that $\forall(c,s)(j) \in \Gamma^*$. Let $([k],Q') \in [\![c]\!]$, so by induction hypothesis, $c(k) \in \Gamma^*$. By the way we interpret binary relations in $\mathcal{N}$, $[\![s]\!](([j],Q),([k],Q'))$. This for all $([k],Q') \in [\![c]\!]$ shows that $([j],Q) \in [\![\forall(c,s)]\!]$.

$\exists(c,s)$ Suppose that $\exists(c,s) \in Sub(\Gamma)$, so that $c \in Sub(\Gamma)$ as well. Let $([j],Q) \in [\![\exists(c,s)]\!]$. Let k and Q' be such that $[\![c]\!]([k],Q')$ and $[\![s]\!](([j],Q),([k],Q'))$. By induction hypothesis, $c(k) \in \Gamma^*$. First, let us consider the case when $Q' = \forall$. Let b be such that Γ^* contains $b(k)$ and $\forall(b,s)(j)$. Using $(\forall E)$, we have $\Gamma^* \vdash \exists(c,s)(j)$. And as Γ^* is closed under deduction, $\exists(c,s)(j) \in \Gamma^*$ as desired. The more interesting case is when $Q' = \exists$, so that for some $j_* \equiv j$ and $k_* \equiv k$, Γ^* contains $s(j_*,k_*)$. Since $c(k)$ and $k \equiv k_*$, we have $c(k_*) \in \Gamma^*$. Then using $(\exists I)$, we see that $\exists(c,s)(j_*) \in \Gamma^*$. Since $j \equiv j_*$, once again we have $\exists(c,s)(j) \in \Gamma^*$.

Conversely, suppose that $\exists(c,s)(j) \in \Gamma^*$. By the Henkin property, let k be such that $c(k)$ and $s(j,k)$ belong to Γ^*. Then $[\![s]\!](([j],Q),([k],\exists))$, and by induction hypothesis, $[\![c]\!](k)$. Hence $([j],Q) \in [\![\exists(c,s)]\!]$.

<u>$\forall(c,a)$</u> Suppose that $\forall(c,a) \in \mathit{Sub}(\Gamma)$, so that $c \in \mathit{Sub}(\Gamma)$ as well. We prove that

$$[\![\forall(c,a)]\!] \quad = \quad \{([j],Q) : \forall(c,a)(j) \in \Gamma^*\}.$$

The first part argument is the left-to-right inclusion. It is exactly the same as what we saw above for the sentences of the form $\forall(c,s)$.

In the other direction, suppose that $\forall(c,a)(j) \in \Gamma^*$; we show that $([j],Q) \in [\![\forall(c,a)]\!]$. For this, let $([k],Q') \in [\![c]\!]$. By induction hypothesis, $c(k) \in \Gamma^*$. We must verify that if $\forall(b,a)(k) \in \Gamma^*$, then also $\forall(b,a)(j) \in \Gamma^*$. This is shown in the derivation below:

$$\dfrac{\dfrac{\dfrac{c(k) \quad \forall(c,a)(j)}{a(j,k)}\ \forall E \qquad \dfrac{[b(x)]^1 \quad \forall(b,a)(k)}{a(k,x)}\ \forall E}{a(j,x)}\ \text{trans}}{\forall(b,a)(j)}\ \forall I^1$$

Since Γ^* is closed under deduction, we see that indeed $\forall(b,a)(j) \in \Gamma^*$. Going on, we see from the structure of $\mathcal{N}$ that $[\![s]\!](([j],Q),([k],Q'))$. This for all $([k],Q') \in [\![c]\!]$ shows that $([j],Q) \in [\![\forall(c,a)]\!]$.

<u>$\exists(c,a)$</u> Suppose that $\exists(c,a) \in \mathit{Sub}(\Gamma)$, so that $c \in \mathit{Sub}(\Gamma)$ as well.

Let $([j],Q) \in [\![\exists(c,a)]\!]$. Let k and Q' be such that the following two assertions hold: $[\![a]\!](([k],Q')$ and $[\![a]\!](([j],Q),([k],Q'))$. By induction hypothesis, $c(k) \in \Gamma^*$. There are two cases depending on whether $Q' = \forall$ or $Q' = \exists$. The argument for $Q' = \forall$ is the same as the one we saw in our work on sentences $\exists(c,s)$ above. The more interesting case is when $Q' = \exists$. This time, j reaches k. By Lemma 10.8.1, $\exists(c,a)(j) \in \Gamma^*$.

Conversely, suppose that $\exists(c,a)(j) \in \Gamma^*$. By the Henkin property, let k be such that $c(k)$ and $a(j,k)$ belong to Γ^*. The derivation below shows that if $\forall(d,a)(k) \in \Gamma^*$, then $\forall(d,a)(j) \in \Gamma^*$ as well:

$$\dfrac{\dfrac{a(j,k) \qquad \dfrac{[d(x)]^1 \quad \forall(d,a)(k)}{a(k,x)}\ \forall E}{a(j,x)}\ \text{trans}}{\forall(d,a)(j)}\ \forall I^1$$

So $[\![a]\!](([j],Q),([k],\exists))$, and by induction hypothesis, $[\![c]\!](k)$. Hence $([j],Q) \in [\![\exists(c,a)]\!]$.

This completes the induction. $\dashv$

Lemma 10.8.4 $\mathcal{N} \models \Gamma$.

Proof We check the sentence types in turn, using Lemma 10.8.3 without mention.

First, let Γ contain the sentence $\forall(b,c)$. Then b and c belong to $\mathit{Sub}(\Gamma)$. Let $([j],Q) \in [\![b]\!]$, so that $b(j) \in \Gamma^*$. We have $d(j) \in \Gamma^*$ using $(\forall E)$. This for all $([j],Q)$ shows that $\mathcal{N} \models \forall(b,c)$.

Second, let $\exists(c,d) \in \Gamma$. By the Henkin property, let j be such that both $c(j)$ and $d(j)$ belong to Γ^*. The element $([j],\forall)$ shows that $[\![c]\!] \cap [\![d]\!] \neq \emptyset$. That is, $\mathcal{N} \models \exists(c,d)$.

Continuing, consider a sentence $b(j) \in \Gamma$. As $b \in Sub(\Gamma)$, we have $([j], \exists) \in [\![b]\!]$, so that $\mathcal{N} \models b(j)$.

The work for sentences of the forms $s(j,k)$ and $\overline{s}(j,k)$ was done in Lemma 10.6.2.

The most intricate part of this proof concerns sentences $a(j,k), \overline{a}(j,k) \in \Gamma$. Recall that we are dealing in this result with sentences of $\mathcal{L}$, and so j and k are constant symbols of that language. Also recall that $[\![j]\!] = ([j], \exists)$, and similarly for k.

Consider sentences in Γ of the form $a(j,k)$. It is easy to see that if $\exists(c,a)(k)$ belongs to Γ, then so does $\exists(c,a)(j)$. (See the $\exists(c,a)$ case in Lemma 10.8.3.) From this it follows easily that $[\![a]\!]([\![j]\!], [\![k]\!])$. And so $\mathcal{N} \models a(j,k)$ in this case.

We conclude with the consideration of a sentence in Γ of the form $\overline{a}(j,k)$. We wish to show that $\mathcal{N} \models \overline{a}(j,k)$. Suppose towards a contradiction that $\mathcal{N} \models a(j,k)$. Then we have $[\![a]\!](([j], \exists), ([k], \exists))$. There are two possibilities, corresponding to the alternatives in the semantics of a. The first is when there is a set term c such that Γ^* contains $c(k)$ and $\forall(c,a)(j)$. Using $(\forall E)$, Γ^* then contains $a(j,k)$. But recall that Γ contains $\overline{a}(j,k)$. So in this alternative, $\Gamma^* \supseteq \Gamma$ is inconsistent. In the second alternative, j reaches k by a chain of $\equiv$ and a statements. By Lemma 10.8.1, $a(j,k) \in \Gamma^*$. So Γ^* is inconsistent, and we have our contradiction. $\dashv$

Once again, this gives us the finite model property for $\mathcal{L}(adj)$. The result is not interesting from a complexity-theoretic point of view, since we already could see from Lutz and Sattler [**?**] that the logic had an ExpTime satisfiability problem.

11 Complexity Results

In this section, we study the computational complexity of the logical systems with which we have been concerned. We are mainly interested in the complexity of the *consequence relation*, that is

$$\{(\Gamma, \varphi) : \Gamma \text{ is a finite set, and } \Gamma \vdash \varphi\}.$$

Recall that we defined *syllogistic proof systems* in Section 9.1. Derivation relations defined by direct proof-systems are easily seen to have polynomial-time complexity.

Lemma 11.0.5 *Let $\mathcal{F}$ be a syllogistic fragment, and $\mathcal{X}$ a finite set of syllogistic rules in $\mathcal{F}$. The problem of determining whether $\Theta \vdash_{\mathcal{X}} \theta$, for a given set of $\mathcal{F}$-formulas Θ and $\mathcal{F}$-formula θ, is in* PTime.

Proof Let Σ be the set of all atoms (unary or binary) occurring in $\Theta \cup \{\theta\}$, together with one additional binary atom r. We first observe that, if there is a derivation of θ from Θ using the rules $\mathcal{X}$, then there is such a derivation involving only the atoms occurring in Σ. For, given any derivation of θ from Θ, uniformly replace any unary atom that does not occur in $\Theta \cup \{\theta\}$ with one that does. Similarly, uniformly replace any binary atom which does not occur in $\Theta \cup \{\theta\}$ with one which does (or with r in case $\Theta \cup \{\theta\}$ contains no binary atoms). This process obviously leaves us with a derivation of θ from Θ, using the rules $\mathcal{X}$.

To prove the lemma, let the the total number of symbols occurring in $\Theta \cup \{\theta\}$ be n. Certainly, $|\Sigma| \leq n$. Let $\mathcal{X}$ comprise k_1 proof-rules, each of which contains at most k_2 atoms (unary or binary). The number of rule instances involving only atoms in Σ is bounded by $p(n) = k_1 n^{k_2}$. Hence, we need never consider derivations with 'depth' greater than $p(n)$. Let Θ_i be the set of formulas involving only the atoms in Σ, and derivable from Θ using a derivation of depth i or less ($0 \leq i \leq p(n)$). Evidently, $|\Theta_i| \leq |\Theta| + p(n)$. It is then straightforward to compute the successive Θ_i in total time bounded by a polynomial function of n. $\dashv$

Theorem 11.0.6 (McAllester and Givan [?]) *The satisfiability problem for a sequent Γ in $\mathcal{RC}$ is* NPTime*-complete. (Thus the validity problem is co-*NPTime*-complete.)*

Proof It follows from the proof of Theorem 6.3.4, that if Γ is a satisfiable theory in $\mathcal{RC}$, then Γ has a model whose size is at most $2n$, where n is the number of positive set terms in the language. (This bound is independent of Γ.) It follows that satisfiability is in NPTime. The main work is in showing the NPTime-hardness.

Our proof is a small variation on the original argument. We use a reduction from the *monotone exactly-1 3SAT* problem. This problem is defined as follows. We are given a conjunction of 3-CNF clauses without negation, so each clause is of the form $U \vee V \vee W$. The problem is to find a truth assignment f to the variables making one variable in each clause T and the other two variables F. We call this a 1-*valued* assignment on $\mathcal{S}$. This problem was shown to be NPTIME-complete in Schaefer [**?**].

Consider a sentence $\mathcal{S}$ which is a conjunction of clauses, each of which is a disjunction of three variables without negation. We define an $\mathcal{RC}$-theory $\Gamma = \Gamma(\mathcal{S})$ via two clauses below. It uses unary atoms which correspond to the variables of $\mathcal{S}$: we use u to correspond with U, etc. Γ also uses a number of new unary and binary atoms. It is defined as follows:

(i) For each clause of $\mathcal{S}$, say $c \equiv U \vee V \vee W$, add to Γ the seven sentences

$$\begin{array}{ll} \forall(x_c, \forall(u, r_c^1)) & \forall(z_c, \forall(w, r_c^3)) \\ \forall(\exists(u, r_c^1), y_c) & \forall(\exists(w, r_c^3), a_c) \\ \forall(y_c, \forall(v, r_c^2)) & \exists(x_c, \overline{a_c}) \\ \forall(\exists(v, r_c^2), z_c) & \end{array}$$

Here x_c, y_c, z_c and a_c are new unary atoms, and r_2^1, r_c^2, and r_c^3 are new binary atoms.

(ii) Let P and Q be any two distinct variables which occur together in some clause c. Then add to Γ the sentence

$$\forall(\forall(p, r_{p,q}), \exists(q, r'_{p,q}))$$

Here $r_{p,q}$ and $r'_{p,q}$ are new binary atoms.

So if $\mathcal{S}$ has k clauses, then the first point will add $4k$ new unary atoms and $3k$ new binary atoms. The second clause will add at most $2 \cdot \binom{3k}{2} < 18k^2$ new binary atoms. We assume that all of the atoms listed are distinct.

The main claim is that $\mathcal{S}$ has a 1-valued assignment iff Γ is satisfiable. In one direction, assume that $\mathcal{M} \models \Gamma$. Define a truth assignment f by $f(U) = \mathsf{F}$ iff $[\![u]\!] \neq \emptyset$. Consider a clause $c \equiv U \vee V \vee W$ of $\mathcal{S}$. If $f(U) = f(V) = f(W) = \mathsf{F}$, then $[\![u]\!]$, $[\![v]\!]$, and $[\![w]\!]$ are all non-empty. By the first six points in (1), $[\![x_c]\!] \subseteq [\![y_c]\!] \subseteq [\![z_c]\!] \subseteq [\![a_c]\!]$. But this contradicts the last point in (1). Thus we know that at least one variable in c is assigned the value T by f. We claim that only one variable can be T. For suppose towards a contradiction that (for example) $f(U) = f(V) = \mathsf{T}$. Then $[\![u]\!] = [\![v]\!] = \emptyset$. So $[\![\forall(u, r_{p,q})]\!] = M$ and $[\![\exists(v, r'_{p,q})]\!] = \emptyset$. By the sentence in (2), M is empty. But this is impossible, since as soon as $\mathcal{S}$ has at least one clause, Γ has an existential assertion via (1). In this way, f is 1-valued on $\mathcal{S}$.

We conclude by checking the converse assertion. Suppose f is 1-valued on $\mathcal{S}$. We must find a model $\mathcal{M} \models \Gamma$. Let M be the set of variables U such that $f(U) = \mathsf{F}$. For a variable X, define $[\![x]\!] = \emptyset$ if $f(X) = \mathsf{T}$, and $[\![x]\!] = \{x\}$ if $f(X) = \mathsf{F}$. We still need to define the interpretations of all of the binary atoms, and all of the other unary atoms. Suppose that P and Q are distinct variables which happen to belong to the same clause. We know that either $f(P) = \mathsf{F}$ or $f(Q) = \mathsf{F}$ (or both). In the first case, set $[\![r_{p,q}]\!] = \emptyset$ so that $[\![\forall(p, r_{p,q})]\!] = \emptyset$. (The interpretation of $r'_{p,q}$ is arbitrary in this case; we no longer mention this point.) This makes $\mathcal{M} \models \forall(\forall(p, r_{p,q}), \exists(q, r'_{p,q}))$. In the second case, $[\![r'_{p,q}]\!] = M \times M$, so that $[\![\exists(q, r'_{p,q})]\!] = M$. In this way, the sentence in (2) holds in $\mathcal{M}$.

Finally, we consider the sentences in (1). If $f(U) = \mathsf{T}$, $f(V) = \mathsf{F}$, and $f(W) = \mathsf{F}$, then we already have $[\![u]\!] = \emptyset$, $[\![v]\!] = \{v\}$, and $[\![w]\!] = \{w\}$. We set $[\![x_c]\!] = M$, $[\![y_c]\!] = \emptyset$, $[\![z_c]\!] = \emptyset$, $[\![a_c]\!] = \{w\}$, $[\![r_c^1]\!] = M \times M$, $[\![r_c^2]\!] = \emptyset$, and $[\![r_c^3]\!] = \{(w, w)\}$.

If $f(U) = \mathsf{F}$, $f(V) = \mathsf{T}$, and $f(W) = \mathsf{F}$, set $[\![x_c]\!] = M$, $[\![y_c]\!] = M$, $[\![z_c]\!] = \{w\}$, $[\![a_c]\!] = \{w\}$, $[\![r_c^1]\!] = M \times M$, $[\![r_c^2]\!] = \emptyset$, and $[\![r_c^3]\!] = \{(w, w)\}$.

If $f(U) = \mathsf{F}$, $f(V) = \mathsf{F}$, and $f(W) = \mathsf{T}$, set $[\![x_c]\!] = M$, $[\![y_c]\!] = M$, $[\![z_c]\!] = M$, $[\![a_c]\!] = \emptyset$, $[\![r_c^1]\!] = M \times M$, $[\![r_c^2]\!] = M \times M$, and $[\![r_c^3]\!] = \emptyset$.

In all cases, the resulting model $\mathcal{M}$ satisfies all sentences in (1), hence all sentences in Γ. ⊣

Lemma 11.0.7 (Pratt-Hartmann and Moss [?]) *The problem of determining the validity of a sequent in $\mathcal{R}^\dagger$ is* EXPTIME*-hard.*

Proof The logic K^U is the basic modal logic K together with an additional modality U (for "universal"), whose semantics are given by the standard relational (Kripke) semantics, plus

$$\models_w U\varphi \text{ if and only if } \models_{w'} \varphi \text{ for all worlds } w'.$$

The satisfiability problem for K^U is EXPTIME-hard. (The proof is an easy adaptation of the corresponding result for propositional dynamic logic; see, e.g. Harel *et al.* [?]: 216 ff.) It suffices, therefore, to reduce this problem to satisfiability in $\mathcal{R}^\dagger$. Let φ be a formula of K^U.

We first transform φ into an equisatisfiable set of formulas $T_\varphi \cup S_\varphi$ of first-order logic; then we translate the formulas of $T_\varphi \cup S_\varphi$ into an equisatisfiable set of $\mathcal{R}^\dagger$-formulas. To simplify the notation, we shall take unary atoms (in $\mathcal{R}^\dagger$) to be unary predicates (in first-order logic); similarly, we take binary atoms to do double duty as binary predicates. Let r and e be binary atoms. For any K^U-formula ψ, let p_ψ be a unary atom, and define the set of first-order formulas T_ψ inductively as follows:

$$\begin{aligned}
T_p &= \emptyset \text{ (where } p \text{ is a proposition letter)} \\
T_{\psi\wedge\pi} &= \begin{array}{l} T_\psi \cup T_\pi \cup \{\forall x(p_\psi(x) \wedge p_\pi(x) \to p_{\psi\wedge\pi}(x)), \\ \qquad\qquad \forall x(p_{\psi\wedge\pi}(x) \to p_\psi(x)), \forall x(p_{\psi\wedge\pi}(x) \to p_\pi(x))\} \end{array} \\
T_{\neg\psi} &= T_\psi \cup \{\forall x(p_{\neg\psi}(x) \to \neg p_\psi(x)), \forall x(\neg p_{\neg\psi}(x) \to p_\psi(x))\} \\
T_{\Box\psi} &= \begin{array}{l} T_\psi \cup \{\forall x(p_{\Box\psi}(x) \to \forall y(\neg p_\psi(y) \to \neg r(x,y))), \\ \qquad \forall x(\neg p_{\Box\psi}(x) \to \exists y(\neg p_\psi(y) \wedge r(x,y)))\} \end{array} \\
T_{U\psi} &= \begin{array}{l} T_\psi \cup \{\forall x(p_{U\psi}(x) \to \forall y(\neg p_\psi(y) \to \neg e(x,y))), \\ \qquad \forall x(\neg p_{U\psi}(x) \to \exists y(\neg p_\psi(y) \wedge e(x,y)))\}. \end{array}
\end{aligned}$$

Now let S_φ be the collection of five first-order formulas

$$\exists x(p_\varphi(x) \wedge p_\varphi(x)), \qquad \forall x(\pm p_\varphi(x) \rightarrow \forall y(\pm p_\varphi(y) \rightarrow e(x,y))).$$

(Although the first formula looks like it has a redundant conjunct, we state it in this way only to make our work below a little easier.) We claim that the modal formula φ is satisfiable if and only if the set of first-order formulas $T_\varphi \cup S_\varphi$ is satisfiable. For let $\mathcal{M}$ be any (Kripke) model of φ over a frame (W, R). Define the first-order structure $\mathcal{M}$ with domain W, by setting $r^{\mathcal{M}} = R$, $e^{\mathcal{M}} = A^2$, and $p_\psi^{\mathcal{M}} = \{w \mid \mathcal{M} \models_w \psi\}$, for any subformula ψ of φ. It is then easy to check that $\mathcal{M} \models T_\varphi \cup S_\varphi$. Conversely, suppose $\mathcal{M} \models T_\varphi \cup S_\varphi$. We build a Kripke structure $\mathcal{M}$ over the frame $(A, r^{\mathcal{M}})$ by setting, for any proposition letter o mentioned in φ, $\mathcal{M} \models_a o$ if and only if $a \in p_o^{\mathcal{M}}$. A straightforward structural induction establishes that for any subformula ψ of φ, $\mathcal{M} \models_a \psi$ if and only if $a \in p_\psi^{\mathcal{M}}$. The formula $\exists x(p_\varphi(x) \wedge p_\varphi(x)) \in S_\varphi$ then ensures that φ is satisfied in $\mathcal{M}$.

Now, all of the formulas in $T_\varphi \cup S_\varphi$ are of one of the forms

$$\forall x(\pm p(x) \rightarrow \pm q(x)) \qquad \forall x(\pm p(x) \rightarrow \forall y(\pm q(y) \rightarrow \pm r(x,y))) \tag{11.1}$$

$$\exists x(p(x) \wedge p(x)) \qquad \forall x(\pm p(x) \rightarrow \exists y(\pm q(y) \wedge r(x,y))) \tag{11.2}$$

$$\forall x(p(x) \wedge q(x) \rightarrow o(x)). \tag{11.3}$$

Notice that formulas of the forms (11.1) and (11.2) translate (in the obvious sense) directly into the fragment $\mathcal{R}^\dagger$; those of form (11.3), by contrast, do not. The next step is to eliminate formulas of this last type.

Let o^* be a new unary relation symbol. For $\theta \in T_\varphi \cup S_\varphi$ of the form (11.3), let r_θ be a new binary atom, and define R_θ to be the set of formulas

$$\forall x(\neg o(x) \rightarrow \exists z(o^*(z) \wedge r_\theta(x,z))) \tag{11.4}$$

$$\forall x(p(x) \rightarrow \forall z(\neg p(z) \rightarrow \neg r_\theta(x,z))) \tag{11.5}$$

$$\forall x(q(x) \rightarrow \forall z(p(z) \rightarrow \neg r_\theta(x,z))), \tag{11.6}$$

which are all of the forms in (11.1) or (11.2). It is easy to check that $R_\theta \models \theta$. For suppose (for contradiction) that $\mathcal{M} \models R_\theta$ and a satisfies p and q but not o in $\mathcal{M}$. By (11.4), there exists b such that $\mathcal{M} \models r_\theta[a,b]$. If $\mathcal{M} \not\models p[b]$, then (11.5) is false in $\mathcal{M}$; on the other hand, if $\mathcal{M} \models p[b]$, then (11.6) is false in $\mathcal{M}$. Thus, $R_\theta \models \theta$ as claimed. Conversely, if $\mathcal{M} \models \theta$, expand $\mathcal{M}$ to a structure $\mathcal{M}'$ by interpreting o^* and r_θ as follows:

$$\begin{aligned} (o^*)^{\mathcal{M}} &= A \\ r_\theta^{\mathcal{M}} &= \{\langle a,a\rangle \mid \mathcal{M} \not\models o[a]\}. \end{aligned}$$

We check that $\mathcal{M}' \models R_\theta$. Formula (11.4) is true, because $\mathcal{M}' \not\models o[a]$ implies $\mathcal{M}' \models r_\theta[a,a]$. Formula (11.5) is true, because $\mathcal{M}' \models r_\theta[a,b]$ implies $a = b$. To see that Formula (11.6) is true, suppose $\mathcal{M}' \models q[a]$ and $\mathcal{M}' \models p[b]$. If $a = b$, then $\mathcal{M} \models o[a]$ (since $\mathcal{M}' \models \theta$); that is, either $a \neq b$ or $\mathcal{M} \models o[a]$. By construction, then, $\mathcal{M}' \not\models r_\theta[a,b]$.

Now let T_φ^* be the result of replacing all formulas θ in T_φ of form (11.3) with the corresponding trio R_θ. (The binary atoms r_θ for the various θ are assumed to be distinct;

however, the same unary atom o^* can be used for all θ.) By the previous paragraph, $T^*_\varphi \cup S_\varphi$ is satisfiable if and only if $T_\varphi \cup S_\varphi$ is satisfiable, and hence if and only if φ is satisfiable. But $T^*_\varphi \cup S_\varphi$ is a set of formulas of the forms (11.1) and (11.2), and can evidently be translated into a set of $\mathcal{R}^\dagger$-formulas satisfied in exactly the same structures. Moreover, this set can be computed in time bounded by a polynomial function of $\|\varphi\|$. This completes the reduction. $\dashv$

We note the following fact. (We omit a detailed proof, since subsequent developments do not hinge on this result.)

Lemma 11.0.8 *The problem of determining the validity of a sequent in $\mathcal{RC}^\dagger$ is in* ExpTime.

Proof Trivial adaptation of Pratt-Hartmann [**?**], Theorem 3, which considers a fragment obtained by adding relative clauses to the relational syllogistic. $\dashv$

Theorem 11.0.9 *The validity problem for $\mathcal{R}^\dagger$ and $\mathcal{RC}^\dagger$ are* ExpTime*-complete.*

Proof Lemmas 11.0.7 and 11.0.8. $\dashv$

Corollary 11.0.10 *There exists no finite set $\mathcal{X}$ of syllogistic rules in either $\mathcal{R}^\dagger$ or $\mathcal{RC}^\dagger$ such that $\vdash_\mathcal{X}$ is both sound and refutation-complete.*

Proof It is a standard result that PTime$\neq$ExpTime. The result is then immediate by Lemmas 11.0.5 and 11.0.7. $\dashv$

Of course, Corollary 11.0.10 leaves open the possibility that there exist *indirect* syllogistic systems that are sound and complete for $\mathcal{R}^\dagger$ and $\mathcal{RC}^\dagger$. To show that there do not, stronger methods are required.

12 Description Logic

13 Categorial Grammar

13.1 Categorial grammar as a syntactic system

Categorial Grammar (CG) is an old tradition in formal grammar and formal semantics. In fact, it is the approach to grammar which is closest to the work that we'll do in these notes.

Basic syntactic categories A categorial grammar always begins with *basic categories*. You should think of these as *simple syntactic categories*. In our linguistic applications, we usually will take N, NP and S, standing for *noun*, *noun phrase*, and *sentence*, respectively. But we could just as well take other basic categories besides these.

Slash categories If C and D are categories, so are $C\backslash D$ and C/D. *It's very important to see the difference between the two slashes!* I personally have names for these:

$\backslash$ *look left*
$/$ *look right*

But they have other names: backslash and slash, over and under. The overall idea is that they are *directional versions* of the usual division notation for fractions:

$$\frac{X}{Y} \text{ corresponds to both } Y\backslash X \text{ and } X/Y$$

The difference is that

$Y\backslash X$ looks for a Y on its left, and then the result is an X
X/Y looks for a Y on its right, and then the result is an X

We sometimes call the categories with slashes *complex categories*, to make it clear that they are not basic categories.

Examples of Categories First, Figure 13.1 shows some categories when the basic ones are S, N, and NP. The complex categories in the figure are going to play the role of *parts of speech* in traditional grammar, such as noun, noun phrase, adjective, etc. So this particular example will be important when we return to our overall topic of inference. For a more artificial example, let the basic categories be S, T, U, V, W, X, and Y. Then some of the categories would be $S/(T\backslash U)$, and also

$$(S/(T\backslash U))\backslash(V/V).$$

syntactic category X	name
S	sentence
N	noun
NP	noun phrase
N/N	adjective
NP\S	verb phrase
(NP\S)\(NP\S)	adverb
(NP\S)/NP	transitive verb
NP/N	determiner

Figure 13.1: Syntactic categories starting from S, N and NP, together with their traditional names.

Lexicons A *lexicon* is a set *Lex* whose elements are called *lexical items*, and with each lexical item a non-empty set of categories. For example, here is a lexicon defined over the basic categories S, T, U, X, and Y:

$$\begin{array}{ll} (a, T/X) & (a, U/Y) \\ (a, S/X) & (a, S/Y) \\ (b, X) & (c, Y) \\ (b, X/T) & (c, Y/U) \end{array} \tag{13.1}$$

We have $Lex = \{a, b, c\}$. We say that a has categories T/X, S/X, U/Y and S/Y. Note that in general a lexical item has more than one category.

For a second example of a lexicon, here is one that uses the basic categories S, N and NP. It also is connected to Figure 13.1 in the sense that the words are of the correct categories. That is, men is a (plural) noun, some is a determiner, walk is a verb phrase, etc.

$$\begin{array}{llll} (\mathsf{men}, \mathrm{N}) & (\mathsf{some}, \mathrm{NP/N}) & (\mathsf{Dana}, \mathrm{NP}) & (\mathsf{teased}, (\mathrm{NP\backslash S})/\mathrm{NP}) \\ (\mathsf{women}, \mathrm{N}) & (\mathsf{no}, \mathrm{NP/N}) & (\mathsf{Kim}, \mathrm{NP}) & (\mathsf{interviewed}, (\mathrm{NP\backslash S})/\mathrm{NP}) \\ (\mathsf{walk}, \mathrm{NP\backslash S}) & (\mathsf{most}, \mathrm{NP/N}) & (\mathsf{smiled}, \mathrm{NP\backslash S}) & (\mathsf{joyfully}, (\mathrm{NP\backslash S})\backslash(\mathrm{NP\backslash S})) \\ (\mathsf{who}, (\mathrm{N\backslash N})/(\mathrm{NP\backslash S})) & (\mathsf{J}, \mathrm{NP}) & (\mathsf{laughed}, \mathrm{NP\backslash S}) & (\mathsf{carefully}, (\mathrm{NP\backslash S})\backslash(\mathrm{NP\backslash S})) \\ (\mathsf{sees}, (\mathrm{NP\backslash S})/\mathrm{NP})) & (\mathsf{M}, \mathrm{NP}) & (\mathsf{cried}, \mathrm{NP\backslash S}) & (\mathsf{excitedly}, (\mathrm{NP\backslash S})\backslash(\mathrm{NP\backslash S})) \\ (\mathsf{every}, \mathrm{NP/N}) & (\mathsf{is\text{-}a}, (\mathrm{NP\backslash S})/\mathrm{N}) & (\mathsf{praised}, (\mathrm{NP\backslash S})/\mathrm{NP}) & \end{array} \tag{13.2}$$

Parsing in CG Let *Lex* be a lexicon. Let *Lex** be the set of finite sequences of lexical items. We use letters like v and w for these sequences, and we call them *words* over the lexicon. We write $w : C$, and we say that *w is of category C* if this assertion can be derived from the rules in Figure 13.2.

For example, let us return to the lexicon in (13.1) and show that $abab : S$. Here is a derivation:

$$\dfrac{a : S/X \qquad \dfrac{b : X/T \qquad \dfrac{a : T/X \quad b : X}{ab : T}}{bab : X}}{abab : S}$$

The leaves are labeled with items from the lexicon, and this is perfectly fine. That is, the leftmost rule in Figure 13.2 says that this is ok. The rest of the derivation above comes from three applications of the / rule in Figure 13.2.

You might try to show that $ababab : S$, but that there is no derivation which would show that $aa : S$.

Turning to our more English-like lexicon in (13.2), here are some derivations of words in the lexicon (that is, sequences of lexical items) of category S:

$$\dfrac{\textsf{Dana}\text{: NP} \qquad \dfrac{\textsf{smiled}\text{: NP}\backslash\text{S} \quad \textsf{joyfully}\text{: (NP}\backslash\text{S)}\backslash\text{(NP}\backslash\text{S)}}{\textsf{smiled joyfully}\text{: NP}\backslash\text{S}}}{\textsf{Dana smiled joyfully}\text{: S}}$$

$$\dfrac{\text{Kim: NP} \qquad \dfrac{\dfrac{\text{criticized: (NP}\backslash\text{S)/NP} \quad \text{Dana: NP}}{\text{criticized Dana: NP}\backslash\text{S}} \qquad \text{carefully: (NP}\backslash\text{S)}\backslash\text{(NP}\backslash\text{S)}}{\text{criticized Dana carefully: NP}\backslash\text{S}}}{\text{Kim criticized Dana carefully: S}}$$

$$\dfrac{\dfrac{\text{F : NP} \qquad \dfrac{\text{and : (NP}\backslash\text{NP)/NP} \qquad \dfrac{\text{B : NP} \qquad \dfrac{\text{and : (NP}\backslash\text{NP)/NP} \quad \text{C : NP}}{\text{and C : }NP\backslash\text{NP}}}{\text{B and C : NP}}}{\text{and B and C : NP}\backslash\text{NP}}}{\text{F and B and C : NP}} \qquad \text{left : NP}\backslash\text{S}}{\text{Farid and Bettina and Cynthia left : S}}$$

Categorial grammars and their languages A *categorial grammar* is a pair $\mathcal{G} = (Lex, C)$, where *Lex* is a lexicon and C is a category. The *language of* $\mathcal{G}$ is the set of words w such that $w : C$.

$$\frac{}{w : C} \qquad \frac{v : B \quad w : B\backslash C}{vw : C}\,\backslash \qquad \frac{v : B/C \quad w : C}{vw : C}\,/$$

Figure 13.2: Categorial parsing. The rule on the left is an axiom coming from the lexicon. The other two rules allow us to eliminate one of the slashes by juxtaposition.

14 Proof-theoretic Semantics